"Quiz" 8 Solutions

MATH 2400

August 1 - August 6, 2012

1. Let $\vec{F} = \langle y, -x, 10(x^2+y^2)z \rangle$, and σ be the surface $z = 2\sqrt{x^2+y^2}$ for $0 \le z \le 9$, with outward orientation. Find the flux of $\vec{F}$ through σ by

   (a) parameterizing the surface using cylindrical coordinates.

   (b) using the function parametrization (or projection).

   (a) The surface $z = 2\sqrt{x^2+y^2}$ has parametrization

$$\vec{r}(z,\theta) = \left\langle \frac{1}{2}z\cos\theta, \frac{1}{2}z\sin\theta, z \right\rangle,\ 0 \le z \le 9,\ 0 \le \theta \le 2\pi.$$

Then

$$\begin{aligned}\vec{r}_z \times \vec{r}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2}\cos\theta & \frac{1}{2}\sin\theta & 1 \\ -\frac{1}{2}z\sin\theta & \frac{1}{2}z\cos\theta & 0 \end{vmatrix} \\ &= \left\langle -\frac{1}{2}z\cos\theta, -\frac{1}{2}z\sin\theta, \frac{1}{4}z \right\rangle \\ &= \frac{1}{4}\langle -2x, -2y, z \rangle .\end{aligned}$$

This normal vector does not point outward, so the vector we want is $\frac{1}{4}\langle 2x, 2y, -z\rangle$ Computing the surface integral, we have

$$\begin{aligned}\iint_\sigma \vec{F}\cdot d\vec{S} &= \int_0^{2\pi}\int_0^9 \langle y, -x, 10(x^2+y^2)z\rangle \cdot \frac{1}{4}\langle 2x, 2y, -z\rangle \, dz\, d\theta \\ &= \int_0^{2\pi}\int_0^9 -\frac{5}{2}(x^2+y^2)z^2 \, dz\, d\theta \\ &= \int_0^{2\pi}\int_0^9 -\frac{5}{8}z^4 \, dz\, d\theta \\ &= 2\pi\left[-\frac{1}{8}z^5\right]_0^9 \\ &= -\frac{9^5}{4}\pi.\end{aligned}$$

(b) Letting $G(x,y,z) = 2\sqrt{x^2+y^2} - z$, we have $\vec{\nabla} G = \left\langle \frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}, -1 \right\rangle$. So,

$$\begin{aligned}
\iint_\sigma \vec{F} \cdot d\vec{S} &= \iint_R \vec{F} \cdot \vec{\nabla} G \, dA \\
&= \iint\limits_{x^2+y^2 \le \frac{81}{4}} \left\langle y,\, -x,\, 20(x^2+y^2)^{\frac{3}{2}} \right\rangle \cdot \left\langle \frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}, -1 \right\rangle dA \\
&= \iint\limits_{x^2+y^2 \le \frac{81}{4}} -20(x^2+y^2)^{\frac{3}{2}} \, dA \\
&= \int_0^{2\pi} \int_0^{\frac{9}{2}} -20 r^4 \, dr \, d\theta \\
&= 2\pi \left[-4r^5 \right]_0^{\frac{9}{2}} \\
&= -\frac{9^5}{4}\pi.
\end{aligned}$$

2. Find the flux of $\vec{F} = \langle z - x - y,\, x - y - z,\, y - x - z \rangle$ through σ, which is the portion of $x^2 + y^2 + z^2 = R^2$ with $y \le 0$, oriented in the negative y direction.

Note that $\hat{n} = \frac{1}{R} \langle x, y, z \rangle$ is a unit normal vector to the surface. So,

$$\begin{aligned}
\iint_\sigma \vec{F} \cdot d\vec{S} &= \iint_\sigma \vec{F} \cdot \hat{n} \, dS \\
&= \iint_\sigma \frac{-(x^2+y^2+z^2)}{R} \, dS \\
&= \iint_\sigma -R \, dS \\
&= -R \cdot (\text{Surface Area of } \sigma) \\
&= -2\pi R^3.
\end{aligned}$$

Alternatively, letting σ_2 be the portion of $y = 0$ for $x^2 + z^2 \le R^2$, oriented in the positive y direction, we have that $\sigma + \sigma_2$ is a closed, outward oriented surface. So, by divergence theorem

$$\begin{aligned}
\iint_{\sigma+\sigma_2} \vec{F} \cdot d\vec{S} &= \iiint_W \operatorname{div}\vec{F} \, dV \\
&- \iiint_W -3 \, dV \\
&= -3 \left(\frac{1}{2} \cdot \frac{4}{3} \pi R^3 \right) \\
&= -2\pi R^3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\iint_{\sigma} \vec{F} \cdot d\vec{S} &= \iint_{\sigma+\sigma_1} \vec{F} \cdot d\vec{S} - \iint_{\sigma_1} \vec{F} \cdot d\vec{S} \\
&= -2\pi R^3 - \iint\limits_{x^2+z^2 \le R^2} \langle z - x - 0,\, x - 0 - z,\, 0 - x - z \rangle \cdot \langle 0, 1, 0 \rangle \, dA \\
&= -2\pi R^3 - \iint\limits_{x^2+z^2 \le R^2} x - z \, dA \\
&= -2\pi R^3 - \int_0^{2\pi} \int_0^R r^2 (\cos\theta - \sin\theta) \, dr \, d\theta \\
&= -2\pi R^3 - 0 \\
&= -2\pi R^3.
\end{aligned}$$

3. Let σ be the boundary of the solid bounded by the surfaces $x^2+y^2=4$, $z=x+4$, $z=-y-4$, oriented outward, and $\vec{F} = \langle -yz^2,\, xz^2,\, z \rangle$.

   (a) Using surface integrals, explicitly compute the flux of $\vec{F}$ through σ.

   (b) Use the Divergence theorem to compute the flux of $\vec{F}$ through σ.

   (a) Let σ_1 be the curved, cylindrical side, σ_2 the top, and σ_3 the bottom. First, note that $\hat{n} = \frac{1}{2} \langle x, y, 0 \rangle$ is the outward pointing, unit normal vector to σ_1. So, the flux is

$$\iint_{\sigma_1} \vec{F} \cdot d\vec{S} = \iint_{\sigma_1} \left\langle -yz^2,\, xz^2,\, z \right\rangle \cdot \frac{1}{2} \langle x, y, 0 \rangle \, dS = \iint_{\sigma_1} 0 \, dS = 0.$$

Using the projection (or function parametrization) we have

$$\begin{aligned}
\iint_{\sigma_2} \vec{F} \cdot d\vec{S} &= \iint\limits_{x^2+y^2 \le 4} \left\langle -yz^2,\, xz^2,\, z \right\rangle \cdot \langle -1, 0, 1 \rangle \, dA \\
&= \iint\limits_{x^2+y^2 \le 4} -y(x+4)^2 + x + 4 \, dA \\
&= 0 + 0 + \iint\limits_{x^2+y^2 \le 4} 4 \, dA \qquad \text{(by symmetry)} \\
&= 16\pi
\end{aligned}$$

Similarly,

$$\begin{aligned}
\iint_{\sigma_3} \vec{F} \cdot d\vec{S} &= \iint\limits_{x^2+y^2 \le 4} \left\langle -yz^2,\, xz^2,\, z \right\rangle \cdot \langle 0, -1, -1 \rangle \, dA \\
&= \iint\limits_{x^2+y^2 \le 4} -x(-y-4)^2 + y + 4 \, dA \\
&= 0 + 0 + \iint\limits_{x^2+y^2 \le 4} 4 \, dA \qquad \text{(by symmetry)} \\
&= 16\pi
\end{aligned}$$

Thus,

$$\iint_\sigma \vec{F} \cdot d\vec{S} = 0 + 16\pi + 16\pi = 32\pi.$$

(b)

$$\begin{aligned}
\iint_\sigma \vec{F} \cdot d\vec{S} &= \iiint_W \operatorname{div}\vec{F}\, dV \\
&= \iint\limits_{x^2+y^2\leq 4} \int_{-y-4}^{x+4} 1\, dz\, dA \\
&= \iint\limits_{x^2+y^2\leq 4} x + y + 8\, dA \\
&= \iint\limits_{x^2+y^2\leq 4} 8\, dA \\
&= 32\pi.
\end{aligned}$$

4. Find the surface area of the portion of $x^2 + y^2 + z^2 = R^2$ above $z = a$, for any $0 \leq a \leq R$.

Letting $G(x, y, z) = x^2 + y^2 + z^2$, we have

$$\vec{\nabla} G = \langle 2x, 2y, 2z \rangle = 2z \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle = 2z\vec{\nabla}\tilde{G}$$

So, the surface area is

$$\begin{aligned}
\iint_\sigma dS &= \iint\limits_{x^2+y^2\leq R^2-a^2} \left|\left|\vec{\nabla}\tilde{G}\right|\right|\, dA \\
&= \iint\limits_{x^2+y^2\leq R^2-a^2} \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}\, dA \\
&= \iint\limits_{x^2+y^2\leq R^2-a^2} \sqrt{\frac{x^2 + y^2 + z^2}{z^2}}\, dA \\
&= \iint\limits_{x^2+y^2\leq R^2-a^2} \frac{R}{z}\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{R^2-a^2}} R\left(R^2 - r^2\right)^{-\frac{1}{2}}\, r\, dr\, d\theta \\
&= 2\pi \left[-R\sqrt{R^2 - r^2}\right]_0^{\sqrt{R^2-a^2}} \\
&= 2\pi R\,(R - a)\,.
\end{aligned}$$