Quiz 7

MATH 2400

July 25, 2012

1. If S is the sphere of radius 16 centered at the origin, parameterize the portion of S contained in the first octant.

   In spherical coordinates, the sphere is given by the equation $\rho = 16$. So, the sphere has parametrization

   $$\vec{r}(u,v) = \langle 16\cos\theta\sin\phi, 16\sin\theta\sin\phi, 16\cos\phi\rangle .$$

   Since we are only want the portion in the first octant, we have the bounds

   $$0 \leq \phi \leq \frac{\pi}{2}, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

2. Calculate the line integral of the vector field $\vec{F} = \left\langle y^2, xy \right\rangle$ along the curve $y = x^3$ from $(0,0)$ to $(1,1)$.

   $$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left\langle (x^3)^2, x(x^3)\right\rangle \cdot \left\langle dx, 3x^2 dx\right\rangle = \int_0^1 4x^6\, dx = \frac{4}{7}.$$

3. Let C be any smooth curve starting at $(1,3)$ and ending at $(0,1)$. Calculate the line integral of $\vec{F}(x,y) = \left\langle 2x+5y, 5x-3y^2\right\rangle$ along C.

   Note that $\vec{F}$ is a conservative vector field. That is, $\vec{F} = \vec{\nabla} f$ on the entire xy-plane, with $f(x,y) = x^2 + 5xy - y^3$. So, by the FTCLI,

   $$\int_C \vec{F}\cdot d\vec{r} = f(0,1) - f(1,3) = -1 - (-11) = 10.$$

4. Calculate the line integral of $\vec{F}(x,y) = \left\langle xy^2 + \arctan(x^2), xe^{y^3} + x^2 y\right\rangle$ along the curve that starts at the origin, travels to $(9,3)$ along $y = \sqrt{x}$, then to $(0,3)$ along $y = 3$, and then back to the origin along $x = 0$.

Since C is closed and $\vec{F}$ is smooth, we can apply Green's theorem:

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \iint_R \frac{\partial}{\partial x}\left(xe^{y^3} + x^2 y\right) - \frac{\partial}{\partial x}\left(xy^2 + \arctan(x^2)\right)\, dA \\
&= \int_0^3 \int_0^{y^2} e^{y^3}\, dx\, dy \\
&= \int_0^3 y^2 e^{y^3}\, dy \\
&= \left[\frac{1}{3} e^{y^3}\right]_0^3 \\
&= \frac{1}{3}\left(e^{27} - 1\right).
\end{aligned}$$