Quiz 5 - Practice

MATH 2400
July 4, 2012

1. Suppose that $w = f(u)$ and that $u = x - y$. Show that,

   (a) $\dfrac{\partial w}{\partial x} = -\dfrac{\partial w}{\partial y}$

   (b) $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial y^2} = -\dfrac{\partial^2 w}{\partial x \partial y}$

   Solution: Notice that $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial x} = \dfrac{\partial w}{\partial u}$ since $\dfrac{\partial u}{\partial x} = 1$.

   Next, observe that $\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial y} = -\dfrac{\partial w}{\partial u}$ since $\dfrac{\partial u}{\partial y} = -1$. This establishes (a).

   For (b), observe that

   $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial u^2}\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 w}{\partial u^2}$ since $\dfrac{\partial u}{\partial x} = 1$.

   Also, $\dfrac{\partial^2 w}{\partial y^2} = -\dfrac{\partial^2 w}{\partial u^2}\dfrac{\partial u}{\partial y} = \dfrac{\partial^2 w}{\partial u^2}$ since $\dfrac{\partial u}{\partial y} = -1$.

   And, $\dfrac{\partial^2 w}{\partial x \partial y} = -\dfrac{\partial^2 w}{\partial u^2}\dfrac{\partial u}{\partial x} = -\dfrac{\partial^2 w}{\partial u^2}$.

2. (a) Find the best quadratic approximation of the function $\cos(xy)$ for (x, y) near $(0, 0)$.

   (b) Use this quadratic approximation to find the following limit:

   $$\lim_{(x,y)\to(0,0)} \frac{\cos(xy) - 1 - xy}{xy}$$

   Solution to (a): The best quadratic approximation for $\cos(xy)$ is the function $Q(x, y) = 1$.

   Solution to (b): Using the quadratic approximation in part (a),

   $$\lim_{(x,y)\to(0,0)} \frac{\cos(xy) - 1 - xy}{xy} = \lim_{(x,y)\to(0,0)} \frac{1 - 1 - xy}{xy} = \lim_{(x,y)\to(0,0)} \frac{-xy}{xy} = -1.$$

3. If the function $u(x,t) = e^{at}\sin(bx)$ satisfies the heat equation $u_t = u_{xx}$, find the relationship between a and b.

   Solution: The equation $u_t = u_{xx}$ implies that $e^{at}(a)\sin(bx) = -e^{at}\sin(bx)(b^2)$ is true for all x. If one chooses an x such that $\sin(bx) \neq 0$, then the above equality implies that $a = -b^2$.

4. Decide if the following function is differentiable at $(0,0)$. Explain your reasoning.

$$f(x,y) = |x| + |y|$$

   Solution: The above function is not differentiable at $(0,0)$ because we will show that $f_x(0,0)$ does not exist. In particular,

   $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ which does not exist.