Quiz 4 Solutions

MATH 2400

June 27, 2012

1. Find the equation of the tangent plane to $f(x, y) = 2x^3y^2 - 5x + y^4 - 8$ at $(-1, 2, 5)$.

$$f_x = 6x^2y^2 - 5 \qquad f_y = 4x^3y + 4y^3$$

$$\begin{aligned} L(x, y) &= f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \\ &= 19(x + 1) + 24(y - 2) + 5 \end{aligned}$$

2. Find the rate of change of $f(x, y) = 3x^2y$ at $(2, -3)$ in the direction $\langle -1, 1\rangle$.

$$\begin{aligned} \vec{\nabla} f &= \left\langle 6xy, 3x^2 \right\rangle \\ \vec{\nabla} f(2, -3) &= \langle -36, 12\rangle \\ f_{\langle -1,1\rangle}(2, -3) &= \vec{\nabla} f(2, -3) \cdot \frac{\langle -1, 1\rangle}{||\langle -1, 1\rangle||} \\ &= 12 \langle -3, 1\rangle \cdot \frac{\sqrt{2}}{2} \langle -1, 1\rangle \\ &= 24\sqrt{2}. \end{aligned}$$

3. Suppose the temperature in an area of space is given by

$$T(x, y, z) = e^{-\left((x-2)^2+2(y+5)^2+3(z-4)^2\right)}.$$

If you are at the point $(4, -2, 3)$, in which direction should you travel for the temperature to increase the fastest? Express your answer as a unit vector.

The temperature, T, will increase the fastest in the direction of $\vec{\nabla} T$.

$$\begin{aligned} \vec{\nabla} T(x, y, z) &= -2T(x, y, z) \langle x - 2, 2(y + 5), 3(z - 4)\rangle \\ \vec{\nabla} T(4, -2, 3) &= 2e^{-25} \langle -2, -6, 3\rangle \\ \frac{\vec{\nabla} T(4, -2, 3)}{\left|\left|\vec{\nabla} T(4, -2, 3)\right|\right|} &= \left\langle -\frac{2}{7}, -\frac{6}{7}, \frac{3}{7} \right\rangle. \end{aligned}$$

4. Let f be a differentiable function of three variables and $w = f(x-y, y-z, z-x)$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Let $s = x - y$, $t = y - z$, $u = z - x$. Then

$$\begin{aligned}
dw &= \frac{\partial w}{\partial s} ds + \frac{\partial w}{\partial t} dt + \frac{\partial w}{\partial s} du \\
&= \frac{\partial w}{\partial s}(dx - dy) + \frac{\partial w}{\partial t}(dy - dz) + \frac{\partial w}{\partial u}(dz - dx) \\
&= \left(\frac{\partial w}{\partial s} - \frac{\partial w}{\partial u}\right) dx + \left(\frac{\partial w}{\partial t} - \frac{\partial w}{\partial s}\right) dy + \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial t}\right) dz \\
\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} &= \left(\frac{\partial w}{\partial s} - \frac{\partial w}{\partial u}\right) + \left(\frac{\partial w}{\partial t} - \frac{\partial w}{\partial s}\right) + \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial t}\right) \\
&= 0.
\end{aligned}$$