Quiz 3 Solutions

MATH 2400

1. Find the equation of the plane that goes through the origin, and is perpendicular to the planes

$$2x + z = 6$$
$$3x + y - 2z = -57.$$

Let $\vec{n_1} = \langle 2, 0, 1 \rangle$, $\vec{n_2} = \langle 3, 1, -2 \rangle$, which are normal vectors to the two given planes. A plane perpendicular to the two given planes will have a normal vector perpendicular to $\vec{n_1}$ and $\vec{n_2}$, which can be found by the cross product:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \langle -1, 7, 2 \rangle .$$

Since the plane goes through the origin, the equation is $-x + 7y + 2z = 0$.

2. Find a vector $\vec{w}$ that bisects the smaller of the two angles formed by $3\hat{\imath} - 4\hat{\jmath}$ and $12\hat{\imath} + 5\hat{\jmath}$.

First, rescale the vectors so that they are unit vectors: $\vec{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, $\vec{v} = \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle$. Letting $\vec{w} = \vec{u} + \vec{v} = \left\langle \frac{99}{65}, \frac{-27}{65} \right\rangle = \frac{9}{65} \langle 11, -3 \rangle$, you can see from the isosceles triangle defined be $\vec{u}$, $\vec{v}$, $\vec{w}$, that $\vec{w}$ bisects the angle.

Alternatively, we can start with $\vec{w}$ as an arbitrary vector $\langle a, b \rangle$. Letting θ be the angle between $\vec{u} = \langle 3, -4 \rangle$ and $\vec{v} = \langle 12, 5 \rangle$, $\vec{w}$ must satisfy

$$\frac{\vec{u} \cdot \vec{w}}{||\vec{u}||\,||\vec{w}||} = \cos\left(\frac{\theta}{2}\right) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||\,||\vec{w}||}$$

Substituting into the equation, we can then solve for $\vec{w}$:

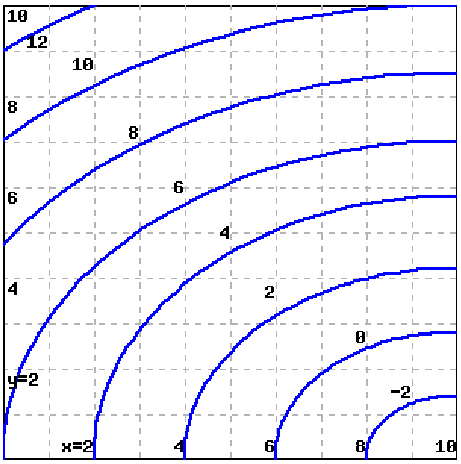
$$\begin{aligned} \frac{3a - 4b}{5\sqrt{a^2 + b^2}} &= \frac{12a + 5b}{13\sqrt{a^2 + b^2}} \\ 39a - 52b &= 60a + 25b \\ -21a &= 77b \\ -\frac{3}{11}a &= b \end{aligned}$$

So, $\vec{w} = \left\langle a, -\frac{3}{11}a \right\rangle = \frac{a}{11} \langle 11, -3 \rangle$, for any $a > 0$ would bisect the angle.

3. Let $f(x,y) = x^4y^5 - \sin(x^2+y^2)$. Compute $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = 4x^3y^5 - 2x\cos(x^2+y^2) \qquad f_y(x,y) = 5x^4y^4 - 2y\cos(x^2+y^2)$$

4. Suppose $f(x,y)$ has the following contour diagram:



Approximate $f_x(3,5)$ and $f_y(3,5)$.

$$f_x(3,5) \approx \frac{f(6,5)-f(3,5)}{\sqrt{(6-3)^2+(5-5)^2}} = \frac{4-6}{3} = -\frac{2}{3}$$
$$f_y(3,5) \approx \frac{f(3,7)-f(3,5)}{\sqrt{(3-3)^2+(7-5)^2}} = \frac{8-6}{2} = 1$$