

Practice Integrals Answers

Setup the following integrals in all reasonable coordinate systems. Compute each quantity in an appropriate coordinate system.

1. The volume the solid bounded by the planes $7x - 8y + 2z = 19$, $5x - y + z = 2$, $y = 3x + 8$, and $x = 4$.

$$\int_{-3}^4 \int_{-\frac{1}{2}(x+5)}^{3x+8} \int_{2-5x+y}^{\frac{1}{2}(19-7x+8y)} 1 \, dz \, dy \, dx = \frac{16807}{8}$$

2. The volume between $z = y^2 + 1$ and $z = 9 - 2x^2 - y^2$.

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{y^2+1}^{9-2x^2-y^2} 1 \, dz \, dy \, dx = \int_0^{2\pi} \int_0^2 \int_{r^2 \sin^2 \theta + 1}^{9-r^2(1+\cos^2 \theta)} r \, dz \, dr \, d\theta = 16\pi$$

3. The mass of the solid that is bounded by the cone $z = \frac{1}{a}\sqrt{x^2 + y^2}$ and the plane $z = b$, and whose density is proportional to the distance from the z -axis.

$$\int_0^{2\pi} \int_0^b \int_0^{az} kr \cdot r \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^{\tan^{-1} a} \int_0^{b \sec \phi} k\rho \sin \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3 b^4 k\pi}{6}$$

4. The volume inside $x^2 + y^2 = R^2$ in the first octant, and below $z = 3x$.

$$\int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{3x} 1 \, dz \, dy \, dx = \int_0^{\frac{\pi}{2}} \int_0^R \int_0^{3r \cos \theta} r \, dz \, dr \, d\theta = R^3$$

5. The volume inside $x^2 + y^2 + z^2 = R^2$ and above $z = x^2 + y^2$.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\sqrt{\frac{-1+\sqrt{1+4R^2}}{2}}} \int_{r^2}^{\sqrt{R^2-r^2}} r \, dz \, dr \, d\theta &= \frac{2}{3}\pi R^3 - \int_0^{2\pi} \int_0^{\frac{-1+\sqrt{1+4R^2}}{2}} \int_{\sqrt{z}}^{\sqrt{R^2-z^2}} r \, dr \, dz \, d\theta \\ &= \frac{\pi R^3}{2} - \frac{\pi}{12} (1 + 4R^2) \left(-1 + \sqrt{1 + 4R^2} \right) \end{aligned}$$

6. The mass of the solid bounded below by $z = \frac{1}{a}\sqrt{x^2 + y^2}$ and above by $x^2 + y^2 + z^2 = R^2$, with density $\delta = 3e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}$.

$$\int_0^{2\pi} \int_0^{\tan^{-1} a} \int_0^R 3e^{\rho^3} \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \left(1 - \frac{1}{\sqrt{1-a^2}} \right) \left(1 - e^{-R^3} \right)$$

7. The volume inside $x^2 + y^2 + z^2 = 2z$ and below $z = 1 + \sqrt{x^2 + y^2}$.

Notice that these are just the sphere $\rho = 1$ and cone $\phi = \frac{\pi}{4}$, shifted up 1 unit. So, we can find the volume inside the standard surfaces in spherical coordinates. or use cylindrical coordinates for the surfaces as stated.

$$\begin{aligned} \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2z-z^2}} r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_1^{\frac{2+\sqrt{2}}{2}} \int_{z-1}^{\sqrt{2z-z^2}} r \, dr \, dz \, d\theta \\ &= \frac{\pi}{3} (2 + \sqrt{2}) \end{aligned}$$

8. The center of mass of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = x^2 + y^2$, with density proportional to the distance from the z -axis.

By symmetry, we know that $\bar{x} = \bar{y} = 0$.

$$\bar{z} = \frac{\int_0^{2\pi} \int_0^1 \int_{r^2}^r z \cdot kr \cdot r \, dz \, dr \, d\theta}{\int_0^{2\pi} \int_0^1 \int_{r^2}^r kr \cdot r \, dz \, dr \, d\theta} = \frac{4}{7}$$

9. The volume inside the cylinders $x^2 + z^2 = R^2$ and $y^2 + z^2 = R^2$.

$$16 \int_0^R \int_0^x \int_0^{\sqrt{R^2 - x^2}} 1 \, dz \, dy \, dx = \frac{16}{3} R^3$$

10. The volume inside the cylinders $x^2 + z^2 = R^2$, $y^2 + z^2 = R^2$ and $x^2 + y^2 = R^2$.

$$16 \int_0^{\frac{\pi}{4}} \int_0^R \int_0^{\sqrt{R^2 - r^2 \cos^2 \theta}} r \, dz \, dr \, d\theta = 8(2 - \sqrt{2}) R^3$$

11. The volume between the surfaces $xy = 1$, $xy = 2$, $xz = 1$, $xz = 3$, $yz = 1$, $yz = 4$.

$$\int_1^2 \int_1^3 \int_1^4 \left| \frac{1}{\begin{vmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{vmatrix}} \right| dw \, dv \, du = \int_1^2 \int_1^3 \int_1^4 \frac{1}{2\sqrt{uvw}} dw \, dv \, du = 4(\sqrt{2} - 1)(\sqrt{3} - 1)$$

12. The volume under $z = \sin(4x^2 + 9y^2)$, above the xy -plane, and inside the elliptical cylinder $4x^2 + 9y^2 = 1$.

$$\int_0^{2\pi} \int_0^1 \sin(r^2) \frac{1}{6} r \, dr \, d\theta = \frac{\pi}{6} (1 - \cos(1))$$