Calculus 3 - Summer 2012
Homework #9
Due 8/8/2012

Written Problems

1. Consider the parameterized surface of revolution

$$\sigma : \vec{r}(z, \theta) = \langle r\cos\theta, r\sin\theta, z\rangle, \ 0 \le \theta \le 2\pi, \ 0 \le z \le 10,$$

where $r = 2\cos(2\pi z) + 5$. Find the flux of $\vec{F} = \langle y^2, x^2, z\rangle$ through σ. *Hint: You may want to consider adjusting the surface in order to use the divergence theorem.*

2. Let $\vec{F} = \left\langle -yz,\ 4xy^2 - xz,\ -\frac{1}{3}x^3z^3 \right\rangle$ and C be the intersection of $z = 13 - 2x^2$ and $z = 4x^2 + 6y^2 - 41$, oriented counterclockwise when viewed from above. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

Presentation Problems

3. Suppose $\vec{F} = \langle f, g, h\rangle$ is a divergence-free field. That is, div $\vec{F} = 0$ everywhere. Define the functions

$$\begin{aligned} u(y,z) &= f(0,y,z) \\ h^x(x,y,z) &= \int_0^x h(t,y,z)\,dt \\ g^x(x,y,z) &= \int_0^x g(t,y,z)\,dt \\ u^y(y,z) &= \int_0^y f(0,t,z)\,dt. \end{aligned}$$

Let $\vec{G} = \langle 0, h^x, u^y - g^x\rangle$.

(a) Show that $\vec{G}$ is a vector potential for $\vec{F}$. That is, $\operatorname{curl}\vec{G} = \vec{F}$. (*Note: You may use the fact that the partial integration defined above commutes with partial differentiation, as long as they are with respect to different variables. That is,* $(h^x)_y = (h_y)^x$.)

(b) Show that if $\vec{H}$ is any other vector potential for $\vec{F}$, then there exists some function ϕ such that $\vec{H} = \vec{G} + \vec{\nabla}\phi$.

4. Let $\vec{F}$ and $\vec{G}$ be vector fields in 3-space such that curl $\vec{F} = \vec{0}$ and div $\vec{G} = 0$ at all points not in a set R. Also, let C be a circle, and σ be a sphere, both not intersecting R. Determine if $\oint_C \vec{F} \cdot d\vec{r}$ and $\iint_\sigma \vec{G} \cdot d\vec{S}$ are guaranteed to be 0 if R is a

(a) point

(b) line

(c) plane

(d) circle

(e) sphere

Would your answers still be correct if C were an arbitrary simple, closed curve and σ were an arbitrary closed surface?

5. Let σ and C be a surface and boundary curve, as in Stokes' theorem, and let W and τ be a solid with boundary surface, as in divergence theorem. Also, let f and g be functions with continuous second-order partial derivatives, and denote $\vec{\nabla}^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = f_{xx} + f_{yy} + f_{zz}$. Prove the following identities:

   (a) $\int_C f\vec{\nabla}g \cdot d\vec{r} = \iint_\sigma (\vec{\nabla}f \times \vec{\nabla}g) \cdot d\vec{S}$

   (b) $\int_C (f\vec{\nabla}g + g\vec{\nabla}f) \cdot d\vec{r} = 0$

   (c) $\iint_\tau f\vec{\nabla}g \cdot d\vec{S} = \iiint_W f\vec{\nabla}^2 g + \vec{\nabla}f \cdot \vec{\nabla}g \, dV$

   (d) $\iint_\tau (f\vec{\nabla}g - g\vec{\nabla}f) \cdot d\vec{S} = \iiint_W f\vec{\nabla}^2 g - g\vec{\nabla}^2 f \, dV$