Calculus 3 - Summer 2012
Homework #8
Due 7/30/2012

Written Problems

1. Let $\vec{F} = \langle g(x,y), h(x,y) \rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

   (a) Compute $\frac{\partial g}{\partial y}$ and $\frac{\partial h}{\partial x}$. Is $\vec{F}$ a gradient field? If so, find a function $f(x,y)$ such that $\vec{F} = \vec{\nabla} f$. On what region does this hold?

   (b) Compute the line integral $\int_{C_R} \vec{F} \cdot d\vec{r}$, where C_R is the circle of radius R, centered at the origin, oriented counterclockwise. Is $\vec{F}$ a conservative vector field?

   (c) Explain why your answers in (a) and (b) are not a contradiction.

2. (a) Use the vector field $\vec{F} = \langle -y, 0 \rangle$ and Green's Theorem to show that the area of the region bounded by $y = 0$, $x = a$, $x = b$, and $y = f(x)$ is $\int_a^b f(x)\, dx$.

   (b) Use the vector field $\vec{F} = \frac{1}{2}\langle -y, x \rangle$ and Green's Theorem to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Suppose that $\vec{F}$ is a continuous, nonzero vector field defined on all of 2-space. Let C be the unit circle centered at the origin and oriented counterclockwise. Suppose that $\int_C \vec{F} \cdot d\vec{r} = 0$. Show that F is perpendicular to C at at least two points.

Presentation Problems

4. For a 2-dimensional vector field $\vec{F} = \langle f(x,y),\, g(x,y) \rangle$ with continuous partials, and curve C with parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle$. Define the following:

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ f(x,y) & g(x,y) & 0 \end{vmatrix} = \langle 0, 0, g_x - f_y \rangle$$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f(x,y),\, g(x,y),\, 0 \rangle = f_x + g_y$$

$$d\vec{s} = \hat{n} ds = \vec{r}\,'_{\perp}(t) dt = \langle y'(t),\, -x'(t) \rangle \; dt$$

   (a) For a closed curve C, bounding a region R, show that $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F}) \cdot \hat{k}\, dA$.

   (b) For a closed curve C, bounding a region R, show that $\oint_C \vec{F} \cdot d\vec{s} = \iint_R \text{div } \vec{F}\, dA$.

5. Consider $\vec{F} = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$.

   (a) Is the domain of $\vec{F}$ path-connected?

   (b) Is $\vec{F}$ path-independent?

   (c) Find the work done by the vector field on a particle moving from $(-1, e)$ to $\left(-e, e^{e^2}\right)$ along $y = e^{x^2}$.

6. Let $\vec{F} = \left\langle e^{x^2} + 4x^2y,\ 144x + \sin y - 9xy^2 \right\rangle$. Find the counterclockwise oriented, simple closed curve, C, with maximal circulation in $\vec{F}$. That is, the curve C where $\oint_C \vec{F} \cdot d\vec{r}$ is maximized.