Calculus 3 - Summer 2012

Homework #5

Due 7/9/2012

Written Problems

1. Prove the following using integration:

   (a) For $a, b > 0$, the triangle with vertices $(0,0), (a,0), (0,b)$ has area $\dfrac{ab}{2!}$.

   (b) For $a, b, c > 0$, the tetrahedron with vertices $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$ has volume $\dfrac{abc}{3!}$.

2. Fix some $R > 0$. Consider the cylinders $x^2 + y^2 = R^2$, $x^2 + z^2 = R^2$, $y^2 + z^2 = R^2$.

   (a) Find the volume of the solid made up of all points satisfying

   $$x^2 + y^2 \leq R^2,\ x^2 + z^2 \leq R^2.$$

   (b) Find the volume of the solid made up of all points satisfying

   $$x^2 + y^2 \leq R^2,\ x^2 + z^2 \leq R^2,\ y^2 + z^2 \leq R^2.$$

Presentation Problems

3. (a) Explain why $\left(\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$.

   (b) Using polar coordinates and improper integrals, show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

4. Evaluate the following integrals. Make sure to show all your work, and give thorough explanations if necessary.

   (a) $\displaystyle\int_0^{\pi}\int_0^{5} 21x^5\cos(x^3 y)\cos^6(\pi x^3)\, dx\, dy$

   (b) $\displaystyle\int_0^1\int_0^{x^3} e^x \sin y\, dy\, dx + \int_0^1\int_{\sqrt{x}}^{1} e^x \sin y\, dy\, dx + \int_0^1\int_{y^2}^{\sqrt[3]{y}} e^x \sin y\, dx\, dy$

   (c) $\displaystyle\iint_R \cos x \sin y\, dA$ where R is the region inside the circle $(x-6)^2 + y^2 = 25$ and outside the square with corners $(3,0)$, $(5,2)$, $(7,0)$, $(5,-2)$.

5. Find the region R that maximizes the integral

   $$\iint_R 144 - 4x^2 - 9y^2\, dA.$$

   Explain carefully why the integral is maximized on R.