Calculus 3 - Summer 2012

Homework #4

Due 7/2/2012

Written Problems

1. Consider the change of variables from rectangular to polar coordinates: $(x, y) = (r\cos\theta, r\sin\theta)$.

   (a) Show that for any differentiable function $w = f(x, y)$,

   $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta, \qquad \frac{\partial w}{\partial \theta} = -\frac{\partial w}{\partial x} r\sin\theta + \frac{\partial w}{\partial y} r\cos\theta.$$

   (b) Show that for any differentiable function $z = f(x, y)$,

   $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

   (c) Show that for any differentiable function $z = f(x, y)$,

   $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

2. Consider the function

   $$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x.y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (a) Show that $f(x, y)$ is differentiable at $(0, 0)$.

   (b) Using the limit definition of partial derivatives, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Presentation Problems

3. For any positive integer p, a function $f(x, y, z)$ is called homogeneous of order p if for any real number t, f satisfies

   $$f(tx, ty, tz) = t^p f(x, y, z).$$

   Show that if f is a homogeneous function of order p, then

   $$\langle x, y, z\rangle \cdot \vec{\nabla} f(x, y, z) = pf(x, y, z)$$

4. Let $f(x, y)$ be differentiable. Carefully show that

   $$f(a, b) = f(0, 0) + \int_0^a f_x(t, 0)\, dt + \int_0^b f_y(a, t)\, dt.$$

5. (a) Find the quadratic approximation of $f(x, y) = e^{-3(x^2+y^2)}$ at $(0, 0)$.

   (b) Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-3(x^2+y^2)} - 1}{x^2 + y^2}$.