Differential Forms

Note: The following contains some advanced topics, not necessarily part of Calc 3. However, Some of the things we will later learn (Jacobian, Divergence, Curl, Chain Rule) will either become easier, or you can gain a deeper understanding with forms. I will be referencing these occasionally in class, so it is helpful to know this, but not required. When I mention them in class, I will only use small bits and pieces, and I will give a quick refresher on what is needed. However, it won't make too much sense without first glancing through this.

The objects dx, dy, dz, df, called differential forms, are not just notation; they do have important meaning in math, but to really know what they are, takes a lot of sophistication. The easiest way to think of these is through vector fields, which we will learn about later on in the course. In short, a vector field assigns a vector to every point. So, for example, the vector field $\langle x, y\rangle$ would have the zero vector at the origin, and radially, outward-pointing vectors at every other point, growing longer as you move further away from the origin.

Every (2D) vector field can be written as a combination of the constant vector fields, $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, denoted more formally as $\frac{\partial}{\partial x}$, $\frac{d}{dy}$. That is. every vector field can be written as $\langle f(x, y), g(x, y)\rangle = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$. The differential forms are defined by

$$dx\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) = f \qquad dy\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) = g.$$

In higher definitions, we have the expected definition

$$dx_k\left(\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i}\right) = \sum_{i=1}^{n} f_i dx_k\left(\frac{\partial}{\partial x_i}\right) = f_k.$$

So, in short, differential forms take in vector fields and spit out functions. So, in some sense, forms are the opposite, or cancel out, vector fields.

Now, we can define the product of 1-forms dx and dy, called a 2-form, by

$$dx \wedge dy\,(u, v) = dx(u)dy(v) - dx(v)dy(u).$$

From this, we get

$$dx \wedge dy\left(f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y}, g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y}\right) = f_1g_2 - f_2g_1 = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

The operation "$\wedge$" is a lot like the cross product. Namely, it is anticommutative, and the distributative law holds. Also, it is associative. In general, for higher order products, it is easiest to use the fact that forms split over sums and differences, and just use the fact that

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\left(f_1\frac{\partial}{\partial x_1}, f_2\frac{\partial}{\partial x_2}, \ldots, f_n\frac{\partial}{\partial x_n}\right) = f_1f_2\cdots f_n.$$

Again, as a reminder, since $\wedge$ is anticommutative ($dx \wedge dy = -dy \wedge dx$), we have that $dx \wedge dx = 0$.

Forms In 3D

In 3 dimensions, with variables $x,\ y,\ z$, every form can be expressed like the following:

$$\begin{aligned} f \qquad\qquad & \text{(0-form)} \\ f\,dx + g\,dy + h\,dz \quad & \text{(1-form)} \\ f\,dx\wedge dy + g\,dy\wedge dz + h\,dz\wedge dx \quad & \text{(2-form)} \\ f\,dx\wedge dy\wedge dz \quad & \text{(3-form)} \end{aligned}$$

There are no higher forms since we only have three dimensions, so a 4-form would have to repeat a variable, making it 0.

Now, we define the derivative, d, which takes in forms, and spits out forms of 1 degree higher. Let f be a function and α be a form just involving $dx,\ dy,\ dz$, and $\wedge$ (That is, like above, but where $f = g = h = 1$ or 0). Then, we define

$$\begin{aligned} d(f) = df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \\ d(f\alpha) &= df\wedge\alpha. \end{aligned}$$

Again, the definition for higher dimensions works exactly how you would expect.
Now, the explicit computation of the derivative of a 1-form:

$$\begin{aligned} & d(f\,dx + g\,dy + h\,dz) \\ &\quad = (f_x\,dx + f_y\,dy + f_z\,dz)\wedge dx + (g_x\,dx + g_y\,dy + g_z\,dz)\wedge dy + (h_x\,dx + h_y\,dy + h_z\,dz)\wedge dz \\ &\quad = 0 + f_y\,dy\wedge dx + f_z\,dz\wedge dx + 0 + g_x\,dx\wedge dy + g_z\,dz\wedge dy + 0 + h_x\,dx\wedge dz + g_y\,dy\wedge dz \\ &\quad = -f_y\,dx\wedge dy + f_z\,dz\wedge dx + g_x\,dx\wedge dy - g_z\,dy\wedge dz - h_x\,dz\wedge dx + h_y\,dy\wedge dz \\ &\quad = (g_x - f_y)\,dx\wedge dy + (h_y - g_z)\,dy\wedge dz + (f_z - h_x) dz\wedge dx. \end{aligned}$$

Note: The above result looks like something you would get by taking the cross product. This fact will be important later.

One can show that, in general, if you are working in any dimension and ω is any form, then

$$d^2\omega = d(d\omega) = 0.$$

This is a VERY important result, although showing it is a bit messy. It is mainly because of this result that we will get all of our theorems in the last third of the course.

Applications

Hodge Star and the Cross Product

Define the Hodge star operator, $\star$, in 3-dimensions, which sends a k-form to a $(3-k)$-form, by

$$\begin{aligned} \star(f) &= f\,dx\wedge dy\wedge dz \\ \star(f\,dx + g\,dy + h\,dz) &= h\,dx\wedge dy + f\,dy\wedge dz + g\,dz\wedge dx \\ \star(f\,dx\wedge dy + g\,dy\wedge dz + h\,dz\wedge dx) &= g\,dx + h\,dy + f\,dz \\ \star(f\,dx\wedge dy\wedge dz) &= f. \end{aligned}$$

Note: $\star^2(\omega) = \star(\star(\omega)) = \omega$.
Now, if we identify a vector $\langle a, b, c\rangle$ with the form $a\,dx + b\,dy + c\,dz$, we can define

$$\vec{u}\times\vec{v} = \star(\vec{u}\wedge\vec{v}).$$

This is in fact the cross product that we already know!

The Chain Rule

Suppose you have a function f and compute its differential, df. If one of the terms is $g\,dx$, and all of the other differentials are independent, then we have that $\frac{\partial f}{\partial x} = g$, by where $\frac{\partial f}{\partial x}$ I mean to be the change of f with respect to the variable x. As a consequence of this, consider a function $f(x, y, t)$ with $x(s, t)$ and $y(t)$. Then, we would have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial t}\,dt \\ &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\,ds + \frac{\partial x}{\partial t}\,dt\right) + \frac{\partial f}{\partial y}\frac{dy}{dt}\,dt + \frac{\partial f}{\partial t}\,dt \\ &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}\,ds + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\right)dt \end{aligned}$$

So, the overall change of f with respect to the variable t would be $\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$. Using other notation, if we consider $F(s, t) = f(x(s, t), y(t), t)$, then this is just $\frac{\partial F}{\partial t}$.

The main note here is that you can compute derivatives using form, that would normally require the chain rule, but now do not need any diagrams to compute.