MATH 2400
Final Exam Review

1. Find an equation for the collection of points that are equidistant to $A(-1,\,5,\,3)$ and $B(6,\,2,\,-2)$.

2. Using a computer, graph a contour plot of $f(x,y) = x^2y - x^3$ and some flow lines of the vector field $\vec{F} = \left\langle 1,\, 3 - 2\frac{y}{x} \right\rangle$. What appears to be true? Prove your conjecture.

3. Find an equation of the plane through the points $(2, 4, -\frac{1}{2})$, $(-1, 2, -\frac{5}{2})$, and $(0, 1, -\frac{3}{2})$.

4. For each of the following limits, calculate the limit if it exits, otherwise show the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

   (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2}$

5. Two tugs are pulling a boat, one pulling in the direction 60° North of East and is half as strong as the other tug. In what direction should the stronger tug pull in order for the boat to move due East? If the weaker tug is pulling with a force of 10 N, what is the net force acting on the boat?

6. Find the acute angle between two diagonals of a cube.

7. Let $f(x,y) = \dfrac{xe^{\sin(x^2y)}}{(x^2+y^2)^{3/2}}$. Compute $f_x(1,0)$. *Note: There is an easy way.*

8. Find the tangent plane to the following surfaces at the given point:

   (a) $z = e^{2y-x} \sin y$ at $(3\pi, \frac{3\pi}{2}, -1)$.

   (b) $\vec{r}(u,v) = \langle u^2, u - v^2, v^2 \rangle$ at $(x,y,z) = (1,-2,1)$.

9. Two legs of a right triangle are measured at 8cm and 15cm, each with a maximum error of 0.2cm. Estimate the maximum error in computing the area and the hypotenuse.

10. Parameterize the line that is tangent to $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

11. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

    (a) the surface defined by $x^2 + z\sin(xyz) = 0$.

    (b) $z = \displaystyle\int_0^{xy^2} e^{t^2}\, dt$.

12. A function $f(x,y)$ is called homogeneous of degree n if for every $t > 0$, $f(tx, ty) = t^n f(x,y)$. Show that if $f(x,y)$ is homogeneous of degree n, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$.

13. Let $z = f(x^2 + \ln y)$. Calculate $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$, and $\dfrac{\partial^2 z}{\partial x \partial y}$.

14. Compute the Taylor series centered at (0,0) of $f(x,y) = \dfrac{1}{1 - x^2 - y^2}$. What is the radius of convergence, and on what set does the series converge?

15. Let $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Is f continuous? Differentiable?

16. Find and classify all critical points of $f(x,y) = \frac{4}{3}x^3 - xy^2 + y$.

17. Find the absolute max/min of $f(x,y) = x^2 + 2y^2 - x$ over the disk $x^2 + y^2 \leq 4$.

18. **(Challenging!)** Consider the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$ with a point P on the surface in the first octant (specifically, where each coordinate is greater than zero). Then the coordinate planes and the tangent plane at P define a tetrahedron in the first octant. Find the point P that will minimize the volume of the resulting tetrahedron.

19. Evaluate the following integrals:

    (a) $\displaystyle\int_0^{16} \int_{\sqrt{y}}^{4} e^{x^3}\, dx\, dy$

    (b) $\displaystyle\iint_R y\, dA$ where R is the region in the first quadrant bounded by $xy = 16$, $y = x$ and $x = 8$.

    (c) $\displaystyle\iiint_G z\, dV$, where G is the solid in the first octant bounded by $x + y = 2$ and $y^2 + z^2 = 4$.

    (d) $\displaystyle\iint_R \frac{\sin(x-y)}{\cos(x+y)}\, dA$ where R is the region bounded by $y = 0$, $y = x$, and $x + y = \frac{\pi}{4}$.

    (e) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx + \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$

    (f) $\displaystyle\int_0^{\sqrt{\frac{3}{2}}} \int_{\frac{1}{\sqrt{3}}x}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^3\, dz\, dy\, dx.$

20. **(Hard)** Three identical cylinders with radii R intersect at the same point at right angles. Find the volume of their intersection.

21. Find the surface area of the portion of the cylinder $x^2 + y^2 = 9$ above the xy-plane, and below $x + y + 3z = 20$.

For the following problems, all closed curves are oriented counterclockwise when viewed from above, and surfaces are oriented outward/upward unless other wise stated.

22. $\displaystyle\int_C x^2\, dx + xy\, dy + z^2\, dz$, $C : \vec{r}(t) = \left\langle \sin t, \cos t, t^2 \right\rangle$, $0 \leq t \leq \pi$.

23. $\displaystyle\int_C 2xz\cos(x^2 z)\, dx + z\, dy + \left(x^2\cos(x^2 z) + y\right)\, dz$, where C is the intersection of $z = 3x^2 + y^3 + 5$ and $y = x^3 - 3$ from $x = 0$ to $x = 1$.

24. $\displaystyle\oint_C (x + y^2)\, dx + (1 + x^2)\, dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y = x^3$.

25. $\displaystyle\oint_C \left\langle x^2 y, \frac{1}{3}x^3, xy \right\rangle \cdot d\vec{r}$ where C is the intersection of $z = y^2 - x^2$ and $x^2 + y^2 = 1$.

26. $\displaystyle\oint_C \langle xy, yz, zx \rangle \cdot d\vec{r}$ where C is the triangle (1,0,0), (0,1,0), (0,0,1).

27. $\displaystyle\oint_C \langle x, y, x^2+y^2\rangle \cdot d\vec{r}$ where C is the boundary of the portion of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

28. $\displaystyle\iint_S \text{curl}\left\langle x\sin^2 z, 3x, z + \tan^{-1}(xy)\right\rangle \cdot d\vec{S}$ where S is the portion of $z = \sqrt{9 - x^2 - y^2}$ inside $x^2 + y^2 = 4$.

29. $\displaystyle\iint_S \text{curl}\left\langle x^2 e^{yz}, y^2 e^{xz}, z^2 e^{xy}\right\rangle \cdot d\vec{S}$ where S is the top half of the sphere $x^2 + y^2 + z^2 = a^2$.

30. Find the flux of $\vec{F}$ through the surface S.

    (a) $\vec{F} = \langle xze^y, -xze^y, z\rangle$, S is the portion of $x + y + z = 1$ in the first octant, oriented downward.

    (b) $\vec{F} = \langle x, y, z\rangle$, S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

    (c) $\vec{F} = \left\langle 3x, xz, z^2\right\rangle$, S is the boundary of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$.

    (d) $\vec{F} = \left\langle x^2 + \sin(yz), y - xe^{-z}, z^2\right\rangle$, S is the boundary of the solid bounded by $x^2 + y^2 = 4$, $x + z = 2$ and $z = 0$.

    (e) $\vec{F} = \langle 2, 5, 3\rangle$, S is the portion of the cone $z = \sqrt{x^2 + y^2}$ inside $x^2 + y^2 = 1$. What if S were instead the portion of $z = x^2 + y^2$ inside $x^2 + y^2 = 1$?

31. For each of the following, determine if the statement is true or false. If true, make sure you can prove it or explain why. if false, give a counterexample.

    (a) For every pair of differentiable functions of one variable, f and g, the line integral $\displaystyle\int_C f(x)\,dx + g(y)\,dy$ is path independent.

    (b) $\vec{F} = \left\langle xy^2, x^2 z\right\rangle$ is an example of a vector field.

    (c) $\displaystyle\int_C \frac{x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy$ is path independent.

    (d) If $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = 0$ for a simple closed curve C, then $\vec{F}$ is conservative.

    (e) If $\nabla^2 f = \nabla \cdot \nabla f \equiv 0$ then $\int_C f_y\,dx - f_x\,dy$ is path independent.

    (f) There is a vector field $\vec{F}$ such that $\text{curl}\vec{F} = \langle 2x, 3yz, -xz^2\rangle$.