Quiz 2 Solutions

MATH 2300-002

January 26, 2010

1. (a) $\displaystyle\int_1^{100} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{100} = 1 - \frac{1}{100} = \frac{99}{100}.$

   (b) $\displaystyle\int_1^{10000} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{10000} = 1 - \frac{1}{10000} = \frac{9999}{10000}.$

   (c) $\displaystyle\int_1^{1000000} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{1000000} = 1 - \frac{1}{1000000} = \frac{999999}{1000000}.$

   (d) $\displaystyle\int_1^{N} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{N} = 1 - \frac{1}{N} = \frac{N-1}{N}.$

   (e) $\displaystyle\lim_{N\to\infty}\int_1^{N} \frac{1}{x^2}\,dx = \lim_{N\to\infty} 1 - \frac{1}{N} = 1.$

   (f) What does this limit represent? This says that the area under $f(x) = \dfrac{1}{x^2}$ to the right of $x = 1$ is 1.

2. Repeat the last problem with $f(x) = \dfrac{1}{\sqrt{x}}$.

   (a) $\displaystyle\int_1^{100} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{100} = 2\sqrt{100} - 2 = 18.$

   (b) $\displaystyle\int_1^{10000} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{10000} = 2\sqrt{10000} - 2 = 198.$

   (c) $\displaystyle\int_1^{1000000} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{1000000} = 2\sqrt{1000000} - 2 = 1998.$

   (d) $\displaystyle\int_1^{N} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{N} = 2\sqrt{N} - 2.$

   (e) $\displaystyle\lim_{N\to\infty}\int_1^{N} \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty} 2\sqrt{N} - 2 = \infty.$

   (f) What does this limit represent? This says that the area under $f(x) = \dfrac{1}{\sqrt{x}}$ to the right of $x = 1$ is infinite.

3. Verify Simpson's rule with one iteration (SIMP(1)) gives the exact answer for the definite integral of every cubic (or lower) polynomial.

Note that for constants a, b and continuous functions f, g we have that $\text{LEFT}(af+bg, n) = a\text{LEFT}(f, n) + b\text{LEFT}(g, n)$, and similarly for the other evaluation functions. So, it is enough to show for $1, x, x^2$ and x^3. Furthermore, a substitution of $u = \dfrac{x-a}{b-a}$ will convert a general integral on $[a, b]$ to one on [0,1]. This will complicate the integrand, but it will still be a cubic on u. So, it is enough to consider $\displaystyle\int_0^1 1\,dx,\ \int_0^1 x\,dx,\ \int_0^1 x^2\,dx,\ \int_0^1 x^3\,dx.$

	ACTUAL	LEFT	RIGHT	MID	TRAP	SIMP
$\int_0^1 1\,dx$	1	1	1	1	$\frac{1+1}{2} = 1$	$\frac{2(1)+1}{3} = 1$
$\int_0^1 x\,dx$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{0+1}{2} = \frac{1}{2}$	$\frac{2\left(\frac{1}{2}\right)+\frac{1}{2}}{3} = \frac{1}{2}$
$\int_0^1 x^2\,dx$	$\frac{1}{3}$	0	1	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\frac{0+1}{2} = \frac{1}{2}$	$\frac{2\left(\frac{1}{4}\right)+\frac{1}{2}}{3} = \frac{1}{3}$
$\int_0^1 x^3\,dx$	$\frac{1}{4}$	0	1	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\frac{0+1}{2} = \frac{1}{2}$	$\frac{2\left(\frac{1}{8}\right)+\frac{1}{2}}{3} = \frac{\frac{1}{4}+\frac{2}{4}}{3} = \frac{1}{4}$

4. Use partial fractions to expand $\dfrac{1}{x^4-1} = \dfrac{1}{(x-1)(x+1)(x-i)(x+i)}$. Your work should involve complex numbers, but your answer should not.

Using the cover-up method, we get

$$\begin{aligned}
\frac{1}{(x-1)(x+1)(x-i)(x+i)} &= \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{1/(4i)}{x+i} + \frac{-1/(4i)}{x-i} \\
&= \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{\frac{x-i-(x+i)}{4i}}{x^2+1} \\
&= \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{\frac{-2i}{4i}}{x^2+1} \\
&= \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{x^2+1}.
\end{aligned}$$