Quiz 1 Solutions

MATH 2300-002

January 19, 2010

1. $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx =$

$$\int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-(x^2)^2}\,2xdx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$$

Geometrically, the integral is the area of the portion of the unit circle in the first quadrant. Thus, we have

$$\int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}.$$

2. $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx =$

Substituting $y = x+1$, we get

$$\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{(y-1)e^{y-1}}{y^2}\,dy = \int y^{-1}e^{y-1}\,dy - \int y^{-2}e^{y-1}\,dy.$$

Applying integration by parts to the first integral with $u = y^{-1}$,

$$\int y^{-1}e^{y-1}\,dy = y^{-1}e^{y-1} + \int y^{-2}e^{y-1}\,dy$$

So, $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx = \int y^{-1}e^{y-1}\,dy - \int y^{-2}e^{y-1}\,dy = y^{-1}e^{y-1} + C = \frac{e^x}{x+1} + C.$

3. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\sin x}{1+x^6}\,dx =$

Note that we are integrating an odd function ($f(-x) = -f(x)$) over an interval of the form $[-a, a]$. So, the integral is 0.

4. Using the substitution $u = \sqrt{x}$,

$$\begin{aligned}\int \frac{\sqrt{x}}{(x-1)}\,dx &= 2\int \frac{u^2}{u^2-1}\,du \\ &= \int 2 + \frac{2}{(u-1)(u+1)}\,du \\ &= 2u + \int \frac{1}{u-1} - \frac{1}{u+1}\,du \\ &= 2u + \ln|u-1| - \ln|u+1| + C \\ &= 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C.\end{aligned}$$

5. First, note that $\displaystyle\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$ and $\displaystyle\int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C$. So, using integration by parts with $u = x$, we have

$$\begin{aligned}\int xe^x \sin x\,dx &= x\frac{e^x}{2}(\sin x - \cos x) - \int \frac{e^x}{2}(\sin x - \cos x)\ dx \\ &= \frac{1}{2}\left[xe^x(\sin x - \cos x) - \int e^x \sin x\,dx + \int e^x \cos x\,dx\right] \\ &= \frac{1}{2}\left[xe^x(\sin x - \cos x) - \frac{e^x}{2}(\sin x - \cos x) + \frac{e^x}{2}(\sin x + \cos x)\right] + C \\ &= \frac{e^x}{2}[x\sin x - x\cos x + \cos x] + C.\end{aligned}$$

6. Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, we have

$$\begin{aligned}I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx &= -\int_{\frac{\pi}{2}}^0 \frac{\sin^n\left(\frac{\pi}{2}-u\right)}{\sin^n\left(\frac{\pi}{2}-u\right) + \sin^n\left(\frac{\pi}{2}-u\right)}\,du \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos^n u}{\cos^n u + \sin^n u}\,du.\end{aligned}$$

So, $I = \dfrac{1}{2}(I + I) = \dfrac{1}{2}\left(\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx\right) = \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{4}$.