Distance and Area - Definite Integrals

MATH 1300-013

In Chapter 2 we motivated derivatives by looking for the velocity of a moving object, given its position, either as a function or at specific times. We used the familiar equation $d = r \times t$ (or rather $v = \frac{d}{t}$). Finding the average velocity between two times was the same as finding the slope between two points. If we assumed the velocity was constant, then this constant was precisely the slope of the line between the points. This implied that finding velocity from position seemed to be the same as finding the slope of a graph, which gave rise to the idea of a tangent line.

In Chapter 5, we shall try doing the opposite and see what results. We will be given the velocity of a moving object, and will try to find the position. Similar to before, we will try finding the position (or rather, the change in position) between two times. Again, using $d = rt$, we have $d = \Delta s \approx v\Delta t$. Again, if we view velocity as constant, then this corresponds to finding the area under $y = v$ over an interval of length Δt. This is the same as the area of a rectangle with height v and base Δt. This implies that finding the position from velocity seems to be the same as finding the area under a graph. Just like the derivative gives us slope, we will have something called the definite integral, which will give us the area under the curve.

1 Area Under Positive Constant Functions

Open our first applet: `http://calculusapplets.com/inttable.html`.

Also, it would be good to "follow along" graphically with the second applet: `http://calculusapplets.com/intgraph.html`.

This table gives us the velocity of a moving object at various times, along with an estimate of how far the object moved between the times, and overall. Throughout these examples, keep the times at $0 \leq t \leq 8$, but we will be varying how often we measure the speed (number of intervals), and how we approximate the change in position (left or right). Denote the number of intervals by n and the left/right approximations of the total distance traveled by L_n and R_n. Now, let's compute some data:

Steady Velocity					
n	1	2	4	8	16
L_n	160	160	160	160	160
R_n	160	160	160	160	160

This is a very boring table. Since the velocity function is constant, this shouldn't be a surprise. The total traveled distance is

$$d = v\Delta t = (20)(8) = 160.$$

2 Area Under Positive Increasing Functions

Now, back to $n = 1$ and to the second example where the object is Speeding Up. The velocity is changing, but since we know how to compute the distance for constant velocity, we will assume the velocity is constant between the points. So, over the interval, we can either choose the left- or right-hand velocity values. Again, let's compute some data:

Speeding Up					
n	1	2	4	8	16
L_n	160	336	408	440	455
R_n	608	560	520	496	483
$R_n - L_n$	448	224	112	56	28
$(R_n - L_n) * n/8$	56	56	56	56	56
$(R_n + L_n)/2$	384	448	464	468	469

$R_{32} - L_{32} = 14$

$R_{2^k} - L_{2^k} = \dfrac{896}{2^k}$

$\lim\limits_{k\to\infty} R_{2^k} - L_{2^k} = 0$

$(R_n - L_n) * \dfrac{n}{8} = v(8) - v(0) \Rightarrow R_n - L_n = (v(8) - v(0))\dfrac{8-0}{n}$

Since the velocity is increasing, then we can expect that the left-hand approximations are too small and the right-hand approximations are too large. That is, $L_n \le d \le R_n$ for each n. This shows that if we use the left or right approximations, we have a bound on the error (E_n). The error for using the left or right would be

$$|E_n| = |d - L_n| = d - L_n \le R_n - L_n = (v(8) - v(0))\frac{8-0}{n} = \frac{448}{n}.$$

$$|E_n| = |d - R_n| = R_n - d \le R_n - L_n = (v(8) - v(0))\frac{8-0}{n} = \frac{448}{n}.$$

If we instead use the average as an approximation, we will get a much more accurate number. This is actually, at the moment, the most accurate as we can get.

In general, if the velocity is always increasing on an interval $[a, b]$, and we approximate with the left or right approximations, we would have

$$|E_n| \leq R_n - L_n = (v(b) - v(a))\frac{b-a}{n}$$

The expression $\dfrac{b-a}{n}$ occurs often enough that it is usually denoted by Δt_n or just Δt. Note that as n becomes large, the error approaches zero. So, as the number of intervals increases, our approximation will approach the actual value. That is, $\lim_{n\to\infty} E_n = 0$ and $\lim_{n\to\infty} L_n = d = \lim_{n\to\infty} R_n$.

For $k > 0$, $\frac{E_{kn}}{E_n} = \frac{1}{k}$. This will show $E_{kn} = \dfrac{1}{k}E_n$.

3 Area "Under" Mixed Positive And Negative Functions

Now, switch to the fourth example, Forwards and Backwards and fill out another table. Make sure to also look at the graph.

Forwards and Backwards					
n	1	2	4	8	16
L_n	80	48	80	108	125
R_n	720	368	240	188	165

Note that when the velocity is negative, the rectangles are below the t-axis and the area is counting as negative. So, what we are actually computing here is not area, but rather what is called "net signed area". In terms of velocity, we are not computing distance traveled, but rather, the change in position. This might seem strange, but since we are in fact looking for the position function, this is actually what we want.

4 Finding Area From An Equation

Finally, lets do an example by hand. For $v(t) = -\dfrac{1}{10}(t-14)(t+18)$, let's compute the change in position from $t = 2$ to $t = 14$ using $n = 4$ intervals. Fill out tables for L_4 and R_4 similar to the table in the applet.

L_4					
t	2	5	8	11	14
$v(t)$	24	20.7	15.6	8.7	0
velocity	24	20.7	15.6	8.7	
distance	72	62.1	46.8	26.1	
total	207				

R_4					
t	2	5	8	11	14
$v(t)$	24	20.7	15.6	8.7	0
velocity	20.7	15.6	8.7	0	
distance	62.1	46.8	26.1	0	
total	135				

Check your numbers by putting the function, endpoints, and number of intervals into the applet under the "You Try!" example.

5 Area and the Definite Integral as a Function

The (net signed) area under the function $y = f(x)$ on the interval $[a, b]$ is denoted be the definite integral, $\int_a^b f(x)\,dx$. We will now use the third applet (`http://calculusapplets.com/riemann.html`) to approximate some definite integrals. You can experiment with various functions/intervals/sizes to see some examples, but are are looking at what the integrals actually are, so we want the number of intervals to be very large, say 10000. You also may want to adjust the window so the shaded area stays within the viewable screen.

We are now going to fill out a table for the area under various functions $f(x) = nx^{n-1}$ on the intervals $[0, b]$ where b is also varying.

$b\backslash x$	1	2	3	4	5
1	1	1	1	1	1
2	2	4	8	16	32
3	3	9	27	81	243
4	4	16	64	256	1024
5	5	25	125	625	3125
b	b	b^2	b^3	b^4	b^5

This gives the impression that $\displaystyle\int_0^b nx^{n-1}\,dx = b^n$, which is in fact true.