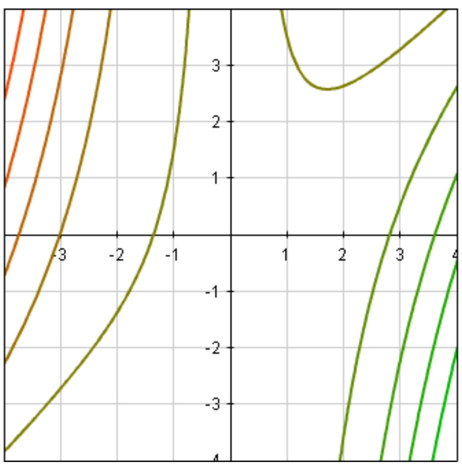
MATH 2400
Final Exam Review Solutions

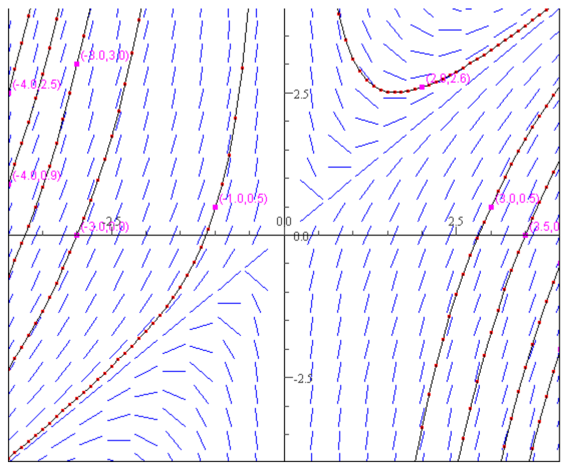
1. Find an equation for the collection of points that are equidistant to $A(-1,\,5,\,3)$ and $B(6,\,2,\,-2)$.

$$\begin{aligned} \left|\left|\overrightarrow{AP}\right|\right|^2 &= \left|\left|\overrightarrow{BP}\right|\right|^2 \\ (x+1)^2+(y-5)^2+(z-3)^2 &= (x-6)^2(y-2)^2+(z+2)^2 \\ x^2+2x+1+y^2-10y+25+z^2-6z+9 &= x^2-12x+36+y^2-4y+4+z^2+4z+4 \\ 14x-6y-10z &= 9. \end{aligned}$$

Alternatively, this is the plane with normal vector $\overrightarrow{AB}$ through the midpoint of A and B.

2. Using a computer, graph a contour plot of $f(x,y)=x^2y-x^3$ and some flow lines of the vector field $\vec{F}=\left\langle 1,\,3-2\frac{y}{x}\right\rangle$. What appears to be true? Prove your conjecture.





Note that it appears that every flow line lies on a level curve. That is, for any flow line C, given by $\vec{r}(t)$, we have that $f(\vec{r}(t))\equiv k$ for some constant k. Note that on C

$$\begin{aligned} \frac{d}{dt}\left(f(\vec{r}(t))\right) &= \vec{\nabla} f\cdot\vec{r}' \\ &= \vec{\nabla} f\cdot\vec{F} \\ &= \left\langle 2xy-3x^2, x^2\right\rangle\cdot\left\langle 1, 3-2\frac{y}{x}\right\rangle \\ &= 0. \end{aligned}$$

Thus, f is constant on any flow line. Therefore, every flow line lies on a level curve. However, note that $x=0$ is a level cure of f, but the vector field is not defined on the entire line. So, not every level curve is a flow line.

Alternatively, we can solve the differential equation

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dY}{dt}}{\frac{dx}{dt}} = \frac{3-2\frac{y}{x}}{1} \\
\frac{dy}{dx} + \frac{2}{x}y &= 3 \\
x^2\frac{dy}{dx} + 2xy &= 3x^2 \\
\frac{d}{dx}\left(x^2y\right) &= 3x^2 \\
x^2y &= x^3 + k \\
f(x,y) = x^2y - x^3 &= k.
\end{aligned}$$

3. Find an equation of the plane through the points $(2,4,-\frac{1}{2})$, $(-1,2,-\frac{5}{2})$, and $(0,1,-\frac{3}{2})$.

   If we compute the vector between successive points, we get $\vec{v}_1 = \langle -3,-2,-2\rangle$, $\vec{v}_2 = \langle 1,-1,1\rangle$. Then

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ -3 & -2 & -2 \\ 1 & -1 & 1 \end{vmatrix} = \langle -4,1,5\rangle$$

   So, one form of the plane would be $-4x + (y-1) + 5\left(z+\frac{3}{2}\right) = 0.$

4. For each of the following limits, calculate the limit if it exits, otherwise show the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin^2 y}{y^2} \cdot \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2} = 1\cdot\lim_{r\to0}\frac{r^4\cos^2\theta\sin^2\theta}{r^2\cos^2\theta+2r^2\sin^2\theta}$$
$$= \lim_{r\to0} r^2\frac{\cos^2\theta\sin^2\theta}{1+\sin^2\theta} \le \lim_{r\to0} r^2\frac{1\cdot\sin^2\theta}{1+\sin^2\theta} \le \lim_{r\to0} r^2\cdot 1 = 0.$$

   Thus, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} = 0.$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

$$y=0: \quad \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,0)\to(0,0)} \frac{x\cdot 0^3}{x^2+0^6} = \lim_{x\to0}\frac{0}{x^2} = \lim_{x\to0} 0 = 0$$
$$x=y^3: \quad \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(y^3,y)\to(0,0)} \frac{y^3\cdot y^3}{(y^3)^2+y^6} = \lim_{y\to0}\frac{y^6}{2y^6} = \lim_{y\to0}\frac{1}{2} = \frac{1}{2} \neq 0.$$

   Thus, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist.

   (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

$$0 \le \left|\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}\right| = \lim_{\rho\to0}\left|\frac{\rho^3\cos\theta\sin\theta\sin^2\phi\cos\phi}{\rho^2}\right| \le \lim_{\rho\to0}\rho = 0$$

   Thus, $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$

5. Two tugs are pulling a boat, one pulling in the direction 60° North of East and is half as strong as the other tug. In what direction should the stronger tug pull in order for the boat to move due East? If the weaker tug is moving at a speed of 10 mph, at what speed is the boat traveling?

   If unattached, the weaker tug would be moving at a speed of $10\sin(60^\circ) = 5\sqrt{3}$ mph North. If the stronger tug is pulling at an angle θ South of East, its unattached vertical speed would be $20\sin\theta$ mph. Since we need the boat to travel due East, we have $20\sin\theta = 5\sqrt{3}$, and so, $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{4}\right) \approx 25.66^\circ$. Then, the horizontal speed of the boat, which is the combined horizontal speed of the tugs, would be $10\cos(60^\circ) + 20\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)\right) = 10\left(\frac{1}{2}\right) + 20\left(\frac{\sqrt{13}}{4}\right) = 5(\sqrt{13}+1) \approx 23.0277$ mph.

6. Find the acute angle between two diagonals of a cube.

   The four vectors along the diagonals are $\langle 1,1,1\rangle, \langle -1,1,1\rangle, \langle -1,-1,1\rangle, \langle 1,-1,1\rangle$. Note that the dot product between any two of these vectors would be ± 1, and the length of each vector is $\sqrt{3}$. So, the acute angle measurement is $\cos^{-1}\left(\frac{1}{\sqrt{3}^2}\right) = \cos^{-1}\left(\frac{1}{3}\right)$.

7. Let $f(x,y) = \dfrac{xe^{\sin(x^2y)}}{(x^2+y^2)^{3/2}}$. Compute $f_x(1,0)$. *Hint: There is an easy way.*

   Let $g(x) = f(x,0) = \dfrac{xe^0}{(x^2+0^2)^{\frac{3}{2}}} = \dfrac{1}{x^2}$. So, $f_x(1,0) = g'(1) = \left.\dfrac{-2}{x^3}\right|_{x=1} = -2$.

8. Find the tangent plane to the following surfaces at the given point:

   (a) $z = e^{2y-x}\sin y$ at $(3\pi, \frac{3\pi}{2}, -1)$.

$$\vec{\nabla} z = \left\langle -e^{2y-x}\sin y, 2e(2y - x\sin y + e^{2y-x}\cos y\right\rangle$$
$$\vec{\nabla} z\left(3\pi, \frac{3\pi}{2}\right) = e^0\langle -(-1), 2+0\rangle = \langle 1,-2\rangle$$

   So, the tangent plane is $z = 1\cdot(x-3\pi) - 2\left(y - \frac{3\pi}{2}\right) - 1 = x - 2y - 1$.

   (b) $\vec{r}(u,v) = \langle u^2, u-v^2, v^2\rangle$ at $(x,y,z) = (1,-2,1)$.

   A plane is defined by a normal vector and a point. Recall $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface, and so, $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface. Note $\vec{r}_u = \langle 2u, 1, 0\rangle$, $\vec{r}_v = \langle 0, -2v, 2v\rangle = 2v\langle 0,-1,1\rangle$, and so, $\vec{r}_u \times \vec{r}_v = 2v\langle 1, -2u, -2u\rangle$. Now, how does $(x,y,z) = (1,-2,1)$ correspond to (u,v)? $x = z = 1$ gives us $u^2 = v^2 = 1$, and so $u = \pm 1, v = \pm 1$. Also, $-2 = y = u - v^2 = u - 1$, which gives us $u = -1$. So, a normal vector is $\langle 1,2,2\rangle$, giving us a tangent plane $(x-1) + 2(y+2) + 2(z-1) = 0$, or $x + 2y + 2z = -1$.

9. Two legs of a right triangle are measured at 8cm and 15cm, each with a maximum error of 0.2cm. Estimate the maximum error in computing the area and the hypotenuse.

$$
\begin{aligned}
A &= \frac{1}{2}xy \\
dA &= \frac{1}{2}y\,dx + \frac{1}{2}x\,dy = A\cdot\left(\frac{dx}{x}+\frac{dy}{y}\right) \\
|\Delta A| &\approx \left|A\left(\frac{\Delta x}{x}+\frac{\Delta y}{y}\right)\right| \le A\left(\frac{|\Delta x|}{x}+\frac{|\Delta y|}{y}\right) \le \frac{1}{2}(8)(15)\left(\frac{.2}{8}+\frac{.2}{15}\right) = 2.3\text{ cm}^2 \\
H &= \sqrt{x^2+y^2} \\
dH &= \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy = \frac{1}{H}(x\,dx+y\,dy) \\
|\Delta H| &\approx \left|\frac{1}{H}(x\,\Delta x + y\,\Delta y)\right| \le \frac{1}{H}\left(x\,|\Delta x| + y\,|\Delta y|\right) = \frac{1}{17}\left(8(.2)+15(.2)\right) = \frac{46}{170} \approx .2706\text{ cm.}
\end{aligned}
$$

10. Parameterize the line that is tangent to $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

Recall that the 3D-gradient of a surface gives a vector perpendicular to the surface. If we cross the two gradients, we will get a vector that is perpendicular to both gradients, and so, tangent to both surfaces. The line starting at our point, pointing along the vector, will be the line we want.

$$
\begin{aligned}
\vec{\nabla}(x^2+y^2-z) &= \langle 2x, 2y, -1\rangle \mapsto \langle -2, 2, -1\rangle \\
\vec{\nabla}(4x^2+y^2+z^2-9) &= \langle 8x, 2y, 2z\rangle \mapsto \langle -8, 2, 4\rangle = 2\langle 4, -1, -2\rangle \\
\langle 2x, 2y, -1\rangle \times \langle 4, -1, -2\rangle &= \langle -5, -8, -6\rangle \\
\vec{r}(t) &= \langle -1, 1, 2\rangle + t\langle 5, 8, 6\rangle .
\end{aligned}
$$

11. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

(a) the surface defined by $x^2 + z\sin(xyz) = 0$.

$$
\begin{aligned}
0 = d(0) = d\left(x^2 + z\sin(xyz)\right) &= 2x\,dx + \sin(xyz)\,dz + \cos(xyz)\,(yz\,dx + xz\,dy + xy\,dz) \\
\left(-\sin(xyz) - xy\cos(xyz)\right)\,dz &= \left(2x + yz\cos(xyz)\right)\,dx + xz\cos(xyz)\,dy \\
dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy &= -\frac{2x + yz\cos(xyz)}{\sin(xyz) + xy\cos(xyz)}\,dx - \frac{xz\cos(xyz)}{\sin(xyz)+xy\cos(xyz)}\,dy
\end{aligned}
$$

(b) $z = \int_0^{xy^2} e^{t^2}\,dt$.

$$
\frac{\partial z}{\partial x} = e^{\left(xy^2\right)^2}\cdot y^2 \qquad \frac{\partial z}{\partial y} = e^{\left(xy^2\right)^2}\cdot 2xy
$$

12. A function $f(x, y)$ is called homogeneous of degree n if for every $t > 0$, $f(tx, ty) = t^n f(x, y)$. Show that if $f(x, y)$ is homogeneous of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

$$
\begin{aligned}
t^n f(x,y) &= f(tx,ty) \\
\frac{d}{dt}\left(t^n f(x,y)\right) &= \frac{d}{dt}\left(f(tx,ty)\right) \\
nt^{n-1}f(x,y) &= f_x(tx,ty)\cdot x + f_y(tx,ty)\cdot y \\
t = 1:\ \ nf(x,y) &= f_x(x,y)\cdot x + f_y(x,y)\cdot y.
\end{aligned}
$$

13. Let $z = f(x^2 + \ln y)$. Calculate $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$, and $\dfrac{\partial^2 z}{\partial x \partial y}$.

$$z_x = 2xf'\left(x^2 + \ln y\right) \qquad z_{xx} = 2f'\left(x^2 + \ln y\right) + 4x^2 f''\left(x^2 + \ln y\right)$$

$$z_y = \frac{1}{y} f'\left(x^2 + \ln y\right) \qquad z_{yy} = \frac{-1}{y^2} f'\left(x^2 + \ln y\right) + \frac{1}{y^2} f''\left(x^2 + \ln y\right)$$

$$z_{yx} = z_{xy} = \frac{2x}{y} f''\left(x^2 + \ln y\right)$$

14. Compute the Taylor series centered at (0,0) of $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$. What is the radius of convergence, and on what set does the series converge?

$$\frac{1}{1 - x^2 - y^2} = \frac{1}{1 - (x^2 + y^2)} = \sum_{n=0}^{\infty} \left(x^2 + y^2\right)^n$$

The series will converge where $\left|x^2 + y^2\right| = x^2 + y^2 < 1$. This is the open disk of radius 1, centered at the origin. Note: You should now be more comfortable with the terminology "radius of convergence."

15. Let $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Is f continuous? Differentiable?

Note that away from the origin, f is a quotient of functions that are differentiable, hence, f is differentiable except possibly at (0,0).

$$\left| \lim_{(x,y)\to(0,0)} f(x, y) \right| = \lim_{r\to 0} \left| \frac{r^2 \cos\theta \sin\theta}{r} \right| = \lim_{r\to 0} r \left|\cos\theta \sin\theta\right| \leq \lim_{r\to 0} r = 0 = f(0, 0).$$

So, f is continuous at (0,0).

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0}{\sqrt{h^2+0^2}} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

Similarly, $f_y(0, 0) = 0$. So, the best linear approximation to f at (0,0) is $L(x, y) = 0(x - 0) + 0(y - 0) + 0 = 0$. So, $E(x, y) = f(x, y) - L(x, y) = f(x, y)$.

$$\lim_{(x,y)\to(0,0)} \frac{E(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2 \cos\theta \sin\theta}{r^2} = \lim_{r\to 0} \cos\theta \sin\theta = \cos\theta \sin\theta \underset{\theta=\frac{\pi}{4}}{=} \frac{1}{2} \neq 0.$$

Thus, f is not differentiable at (0,0).

16. Find and classify all critical points of $f(x, y) = \dfrac{4}{3}x^3 - xy^2 + y$.

$$f_x = 4x^2 - y^2 = 0 \qquad f_y = -2xy + 1 \Rightarrow \left(\pm\frac{1}{2}, \pm 1\right)$$

$$f_{xx} = 8x \qquad f_{xy} = -2y \qquad f_{yy} = -2x$$

$$D(x, y) = (8x)(-2x) - (-2y)^2 = -16x^2 - 4y^2 < 0 \text{ for } (x, y) \neq (0, 0)$$

So, $\left(\pm\frac{1}{2}, \pm 1\right)$ are saddle points.

17. Find the absolute max/min of $f(x,y) = x^2 + 2y^2 - x$ over the disk $x^2 + y^2 \leq 4$.

Note that f is a continuous function, restricted to a compact set. So, we are guaranteed that f will attain a global max and min on the closed disk. Furthermore, they must occur at either critical points on the interior, or on the boundary.

Interior:

$$f_x = 2x - 1 = 0 \qquad f_y = 4y = 0 \Rightarrow \left(\frac{1}{2}, 0\right)$$

We shall apply the method of Lagrange multipliers to the boundary.

$$\begin{aligned}
\langle 2x-1, 4y\rangle &= \lambda \langle 2x, 2y\rangle \\
\frac{2x-1}{2x} &= \lambda = \frac{4y}{2y} \\
y = 0 &\Rightarrow x = \pm 2 \\
y \neq 0 &\Rightarrow \frac{2x-1}{2x} = 2 \\
2x - 1 &= 4x \\
x = -\frac{1}{2} &\Rightarrow y = \pm\sqrt{4 - \left(-\frac{1}{2}\right)^2} = \pm\frac{\sqrt{15}}{2}
\end{aligned}$$

$$f\left(\frac{1}{2}, 0\right) = -\frac{1}{4} < f(2,0) = 2 < f(-2,0) = 6 < f\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right) = \frac{33}{4}$$

18. **(Challenging!)** Consider the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$ with a point P on the surface in the first octant (specifically, where each coordinate is greater than zero). Then the coordinate planes and the tangent plane at P define a tetrahedron in the first octant. Find the point P that will minimize the volume of the resulting tetrahedron.

First, an outline of the solution. We will find the tangent plane at an arbitrary point, and find the volume of the resulting tetrahedron. This then defines a volume function, dependent on the base point. From there, we will apply the method of Lagrange multipliers to the volume function in order to find the minimum.

Consider the point $P(a,b,c)$ with $a,b,c > 0$. Since $\vec{\nabla}\left(\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25}\right) = \left\langle \frac{2}{9}x, \frac{2}{4}y, \frac{2}{25}z \right\rangle$, we have that the tangent plane to the surface at P is

$$\begin{aligned}
0 &= \frac{2}{9}a(x-a) + \frac{2}{4}b(y-b)\frac{2}{25}c(z-c) \\
&= 2\left[\frac{a}{9}x + \frac{b}{4}y + \frac{c}{25}z - \left(\frac{a^2}{9} + \frac{b^2}{4} + \frac{c^2}{25}\right)\right] \\
1 &= \frac{x}{\frac{9}{a}} + \frac{y}{\frac{4}{b}} + \frac{z}{\frac{25}{c}}
\end{aligned}$$

Recall: For $a,b,c > 0$, the volume in the first octant under the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is $\frac{abc}{6}$.

So, the volume of our tetrahedron is $\frac{\frac{9}{a}\cdot\frac{4}{b}\cdot\frac{25}{c}}{6} = \frac{150}{abc}$. Thus, for any point $P(x,y,z)$ satisfying $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$ and $x,y,z > 0$, the volume of the resulting tetrahedron is $V(x,y,z) = \frac{150}{xyz}$.

Notice that we are not working over a closed set, however, as x, y or z approach 0, that volume approaches infinity. So, if we restrict the value to say $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$ and $x, y, z \geq .001$ we are only removing points with a corresponding larger volume, and so, will not change the minimum (should it exist). However, V is continuous on our new set, which is compact. So, we are guaranteed a minimum on that set. To find the minimum, we shall proceed by Lagrange multipliers.

$$\begin{aligned} \vec{\nabla} V(x,y,z) &= \lambda \vec{\nabla}\left(\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25}\right) \\ -V\left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle &= 2\lambda \left\langle \frac{x}{9}, \frac{y}{4}, \frac{z}{25} \right\rangle \\ -\frac{V}{2\lambda} &= \frac{x^2}{9} = \frac{y^2}{4} = \frac{z^2}{25} \end{aligned}$$

Substituting into our constraint, we get

$$3 \cdot \frac{x^2}{9} = 1 \Rightarrow x = \frac{3}{\sqrt{3}} \Rightarrow P(x,y,z) = \left(\frac{3}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right).$$

19. Evaluate the following integrals:

(a) $\int_0^{16} \int_{\sqrt{y}}^4 e^{x^3}\, dx\, dy$

$$\int_0^{16} \int_{\sqrt{y}}^4 e^{x^3}\, dx\, dy = \int_0^4 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^4 x^2 e^{x^3}\, dx = \left[\frac{1}{3} e^{x^3}\right]_0^4 = \frac{e^{64} - 1}{3}.$$

(b) $\iint_R y\, dA$ where R is the region in the first quadrant bounded by $xy = 16$, $y = x$ and $x = 8$.

$$\int_4^8 \int_{\frac{16}{x}}^x y\, dy\, dx = \int_4^8 \frac{1}{2}x^2 - \frac{2^7}{x^2}\, dx = \frac{176}{3}.$$

(c) $\iiint_G z\, dV$, where G is the solid in the first octant bounded by $x + y = 2$ and $y^2 + z^2 = 4$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{2 - r\sin\theta} r\cos\theta \cdot r\, dx\, dr\, d\theta &= \int_0^{\frac{\pi}{2}} \int_0^2 r^2 \cos\theta\,(2 - r\sin\theta)\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{16}{3}\cos\theta - 4\sin\theta\cos\theta\, d\theta = \frac{10}{3}. \end{aligned}$$

(d) $\iint_R \frac{\sin(x-y)}{\cos(x+y)}\, dA$ where R is the region bounded by $y = 0$, $y = x$, and $x + y = \frac{\pi}{4}$.

$$x = u + v \qquad y = v - u$$

$$\left\{\begin{array}{rcl} y &=& 0 \\ y &=& x \\ x+y &=& \frac{\pi}{4} \end{array}\right\} \mapsto \left\{\begin{array}{rcl} u &=& v \\ u &=& 0 \\ v &=& \frac{\pi}{4} \end{array}\right\} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$\iint_R \frac{\sin(x-y)}{\cos(x+y)}\,dA = \int_0^{\frac{\pi}{4}} \int_0^v \frac{\sin u}{\cos v} 2\,du\,dv = 2\int_0^{\frac{\pi}{4}} -1+\frac{1}{\cos v}\,dv = 2\ln(\sqrt{2}+1)-\frac{\pi}{2}.$$

(e) $$\int_0^{\frac{1}{\sqrt{2}}} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx + \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^2 \sqrt{r^2}r\,dr\,d\theta = \frac{14}{3}\pi.$$

(f) $$\int_0^{\sqrt{\frac{3}{2}}} \int_{\frac{1}{\sqrt{3}}x}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^3\,dz\,dy\,dx.$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^2 \left(\rho^2\right)^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \left(\frac{\pi}{2}-\frac{\pi}{6}\right)\left[-\cos\phi\right]_0^{\frac{\pi}{4}} \left[\frac{1}{9}\rho^9\right]_0^2 = \frac{512\pi}{27}\left(2-\sqrt{2}\right).$$

20. **(Hard)** Three identical cylinders with radii R intersect at the same point at right angles. Find the volume of their intersection.

Consider the cylinders $x^2+y^2=R^2$, $x^2+z^2=R^2$, $y^2+z^2=R^2$. Using symmetry, we can restrict out attention to $z \geq 0$, $x \geq y \geq 0$, giving us

$$16 \iint_{\substack{0\leq y\leq x\leq R\\ x^2+y^2\leq R^2}} \sqrt{R^2-x^2}\,dA = 16\int_0^{\frac{\pi}{4}} \int_0^R r\sqrt{R^2-r^2\cos^2\theta}\,dr\,d\theta$$

$$= 16\int_0^{\frac{\pi}{4}} \left[-\frac{1}{3\cos^2\theta}\left(R^2-r^2\cos^2\theta\right)^{\frac{3}{2}}\right]_0^R d\theta$$

$$= 16\int_0^{\frac{\pi}{4}} -\frac{1}{3\cos^2\theta}R^3\sin^3\theta + \frac{R^3}{3\cos^2\theta}\,d\theta$$

$$= \frac{16}{3}R^3 \int_0^{\frac{\pi}{4}} \left(\sec^2 - \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\right) d\theta$$

$$= \frac{16}{3}R^3 \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - \frac{\sin\theta}{\cos^2\theta} + \sin\theta\right) d\theta$$

$$= \frac{16}{3}R^3 \left[\tan\theta - \frac{1}{\cos\theta} - \cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{16}{3}R^3 \left(1-\sqrt{2}-\frac{\sqrt{2}}{2}-0+1+1\right)$$

$$= \frac{16}{3}R^3 \left(3-\frac{3}{2}\sqrt{2}\right) = 8R^3\left(2-\sqrt{2}\right).$$

21. Find the surface area of the portion of the cylinder $x^2+y^2=9$ above the xy-plane, and below $x+y+3z=20$.

$$\begin{aligned}
\vec{r}(\theta, z) &= \langle 3\cos\theta, 3\sin\theta, z\rangle \\
\vec{r}_\theta &= \langle -3\sin\theta, 3\cos\theta, 0\rangle \qquad \vec{r}_z = \langle 0,0,1\rangle \\
||\vec{r}_\theta \times \vec{r}_z|| &= ||\langle 3\cos\theta, 3\sin\theta, 0\rangle|| = \sqrt{9\cos^2\theta + 9\sin^2\theta + 0} = 3 \\
\iint_S dS &= \int_0^{2\pi}\int_0^{\frac{1}{2}(20-3\cos\theta-3\sin\theta)} 3\,dz\,d\theta \\
&= \int_0^{2\pi} 20 - 3\cos\theta - 3\sin\theta\,d\theta \\
&= 40\pi.
\end{aligned}$$

For the following problems, all closed curves are oriented counterclockwise when viewed from above, and surfaces are oriented outward/upward unless other wise stated.

22. $\displaystyle\int_C x^2\,dx + xy\,dy + z^2\,dz,\ C : \vec{r}(t) = \left\langle \sin t, \cos t, t^2\right\rangle,\ 0 \le t \le \pi.$

$$\int_C x^2\,dx + xy\,dy + z^2\,dz = \int_0^\pi \sin^2 t\cdot\cos t + \sin t\cos t(-\sin t) + (t^2)^2(2t)\,dt = \int_0^\pi 2t^5\,dt = \frac{\pi^6}{3}.$$

23. $\displaystyle\int_C 2xz\cos(x^2z)\,dx + z\,dy + \left(x^2\cos(x^2z) + y\right)\,dz$, where C is the intersection of $z = 3x^2 + y^3 + 5$ and $y = x^3 - 3$ from $x = 0$ to $x = 1$.

$$\begin{aligned}
&\int_C 2xz\cos(x^2z)\,dx + z\,dy + \left(x^2\cos(x^2z) + y\right)\,dz \\
&= \int_{(0,-3,-22)}^{(1,-2,0)} \vec{\nabla}\left(yz + \sin(x^2z)\right)\cdot d\vec{r} \\
&= (-2)(0) + \sin(0) - (-3)(-22) - \sin(0) = -66.
\end{aligned}$$

24. $\displaystyle\oint_C (x + y^2)\,dx + (1 + x^2)\,dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y = x^3$.

$$\begin{aligned}
\int_C (x + y^2)\,dx + (1 + x^2)\,dy \underset{\text{Green's}}{=} &\int_0^1\int_{x^3}^{x^2} (2x - 2y)\,dy\,dx \\
= &\int_0^1 2x^3 - x^4 - 2x^4 + x^6\,dx = \frac{3}{70}.
\end{aligned}$$

25. $\displaystyle\oint_C \left\langle x^2y, \frac{1}{3}x^3, xy\right\rangle\cdot d\vec{r}$ where C is the intersection of $z = y^2 - x^2$ and $x^2 + y^2 = 1$.

Let S be the surface $z = y^2 - x^2$ over $x^2 + y^2 \le 1$, which will have boundary curve C.

$$\oint_C \left\langle x^2y, \frac{1}{3}x^3, xy \right\rangle \cdot d\vec{r} \underset{\text{Stokes'}}{=} \iint_S \text{curl} \left\langle x^2y, \frac{1}{3}x^3, xy \right\rangle \cdot d\vec{S}$$

$$= \iint_{x^2+y^2\leq 1} \langle x, -y, 0\rangle \cdot \langle 2x, -2y, 1\rangle \; dA$$

$$= \int_0^{2\pi} \int_0^1 2r^2 \cdot r\, dr\, d\theta = \pi.$$

26. $\oint_C \langle xy, yz, zx\rangle \cdot d\vec{r}$ where C is the triangle (1,0,0), (0,1,0), (0,0,1).

Let S be the flat interior of C, given by $x + y + z = 1$, $0 \leq y \leq 1 - x \leq 1$.

$$\oint_C \langle xy, yz, zx\rangle \cdot d\vec{r} \underset{\text{Stokes'}}{=} \iint_S \text{curl} \langle xy, yz, zx\rangle \cdot d\vec{S}$$

$$= \iint_R \langle -y, -z, -x\rangle \cdot \langle 1, 1, 1\rangle \; dA$$

$$= \iint_R -(x + y + z) \; dA$$

$$= -1 \cdot \iint_R dA = -\frac{1}{2}$$

27. $\oint_C \left\langle x, y, x^2 + y^2\right\rangle \cdot d\vec{r}$ where C is the boundary of the portion of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Let S be the portion of $z = 1 - x^2 - y^2$ inside C.

$$\oint_C \left\langle x, y, x^2 + y^2\right\rangle \cdot d\vec{r} \underset{\text{Stokes'}}{=} \iint_S \text{curl} \left\langle x, y, x^2 + y^2\right\rangle \cdot d\vec{S}$$

$$= \iint_R \langle 2y, -2x, 0\rangle \cdot \langle 2x, 2y, 1\rangle \; dA$$

$$= \iint_R 0 \, dA = 0$$

28. $\iint_S \text{curl} \left\langle x \sin^2 z, 3x, z + \tan^{-1}(xy)\right\rangle \cdot d\vec{S}$ where S is the portion of $z = \sqrt{9 - x^2 - y^2}$ inside $x^2 + y^2 = 4$.

By either applying Stokes' twice or divergence theorem, any surface with same orientation and boundary as S will have the same surface integral. Let S' be the disk $z = \sqrt{5}$ with $x^2 + y^2 \leq 4$.

$$\iint_S \operatorname{curl}\left\langle x\sin^2 z, 3x, z+\tan^{-1}(xy)\right\rangle \cdot d\vec{S} \underset{\text{Divergence}}{=} \iint_{S'} \operatorname{curl}\left\langle x\sin^2 z, 3x, z+\tan^{-1}(xy)\right\rangle \cdot d\vec{S}$$
$$= \iint_{x^2+y^2\le 4} \left\langle \frac{x}{1+x^2y^2}, \frac{-y}{1+x^2y^2}, 3\right\rangle \cdot \langle 0,0,1\rangle \, dA$$
$$= 3\iint_{x^2+y^2\le 4} dA = 12\pi.$$

29. $\displaystyle\iint_S \operatorname{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot d\vec{S}$ where S is the top half of the sphere $x^2+y^2+z^2=a^2$.
Note that S', the disk $z=0$ with $x^2+y^2\le a^2$, has the same boundary curve as S.

$$\iint_S \operatorname{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot d\vec{S} \underset{\text{Divergence}}{=} \iint_{S'} \operatorname{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot d\vec{S}$$
$$= \iint_{x^2+y^2\le a^2} \operatorname{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot \langle 0,0,1\rangle \, dA$$
$$= \iint_{x^2+y^2\le a^2} z\left(y^2e^{xz} - x^2e^{yz}\right) dA$$
$$= \iint_{x^2+y^2\le a^2} 0\cdot\left(y^2e^{0} - x^2e^{0}\right) dA$$
$$= \iint_{x^2+y^2\le a^2} 0\, dA = 0.$$

30. Find the flux of $\vec{F}$ through the surface S.

(a) $\vec{F} = \langle xze^y, -xze^y, z\rangle$, S is the portion of $x+y+z=1$ in the first octant, oriented downward.

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-x} \langle xze^y, -xze^y, z\rangle\cdot\langle -1,-1,-1\rangle \, dy\, dx = \int_0^1\int_0^{1-x} -xze^y + xze^y - z \, dy\, dx$$
$$= \int_0^1\int_0^{1-x} -1+x+y \, dy\, dx = \int_0^1 -(1-x)^2 + \frac{1}{2}(1-x)^2 \, dx = -\frac{1}{6}.$$

(b) $\vec{F} = \langle x, y, z\rangle$, S is the upper half of the sphere $x^2+y^2+z^2=a^2$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{r}\cdot\frac{\vec{r}}{||\vec{r}||}\, dS = \iint_S ||\vec{r}||\, dS = \iint_S a\, dS = a\cdot\frac{1}{2}\cdot 4\pi a^2 = 2\pi a^3.$$

(c) $\vec{F} = \left\langle 3x, xz, z^2\right\rangle$, S is the boundary of the solid bounded by $z = 4-x^2-y^2$ and $z=0$.

$$\iint_S \vec{F}\cdot d\vec{S} \underset{\text{Divergence}}{=} \iint\limits_{x^2+y^2\le 4} \int_0^{4-x^2-y^2} \text{div}\vec{F}\, dz\, dA$$
$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3+2z)r\, dz\, dr\, d\theta$$
$$= 2\pi \int_0^2 r\left(3(4-r^2)+(4-r^2)^2\right)\, dr$$
$$= \frac{136}{3}\pi.$$

(d) $\vec{F} = \left\langle x^2+\sin(yz), y - xe^{-z}, z^2\right\rangle$, S is the boundary of the solid bounded by $x^2+y^2 = 4$, $x+z=2$ and $z=0$.

$$\iint_S \vec{F}\cdot d\vec{S} \underset{\text{Divergence}}{=} \iint\limits_{x^2+y^2\le 4} \int_0^{2-x} 2x+1+2z\, dz\, dA$$
$$= \iint\limits_{x^2+y^2\le 4} (2x+1)(2-x)+(2-x)^2\, dA$$
$$= \iint\limits_{x^2+y^2\le 4} -x^2 - x + 6\, dA$$
$$= \int_0^{2\pi}\int_0^2 -r^3\cos^2\theta - r^2\cos\theta + 6r\, dr\, d\theta$$
$$= \int_0^{2\pi} -4\cos^2\theta - \frac{8}{3}\cos\theta + 12\, d\theta$$
$$= 20\pi.$$

(e) $\vec{F} = \langle 2,5,3\rangle$, S is the portion of the cone $z = \sqrt{x^2+y^2}$ inside $x^2+y^2=1$. What if S were instead the portion of $z = x^2+y^2$ inside $x^2+y^2=1$?
Notice that the cone and the paraboloid can both be continuously deformed to the disk,, $S' : z = 1$, $x^2+y^2\le 1$ and $\text{div}\vec{F} = 0$, so the flux through all three surfaces will be equal.

$$\iint_{S'} \vec{F}\cdot d\vec{S} = \iint\limits_{x^2+y^2\le 1} \langle 2,5,3\rangle\cdot\langle 0,0,1\rangle\, dA = 3\iint\limits_{x^2+y^2\le 1} dA = 3\pi.$$

31. For each of the following, determine if the statement is true or false. If true, make sure you can prove it or explain why. if false, give a counterexample.

(a) For every pair of differentiable functions of one variable, f and g, the line integral $\int_C f(x)\, dx + g(y)\, dy$ is path independent.
Since f and g are differentiable, they have antiderivatives, F, G. Since $\vec{\nabla}(F(x)+G(y)) = \langle f(x), g(y)\rangle$, then $\int_C f(x)\, dx + g(y)\, dy$ is path independent.

(b) $\vec{F} = \left\langle xy^2, x^2x\right\rangle$ is an example of a vector field.
$\vec{F} = \left\langle xy^2, x^2z\right\rangle$ has three inputs, but only two outputs, and so, $\vec{F}$ cannot be a vector field.

(c) $\int_C \frac{x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy$ is path independent.

$$\text{curl}\left\langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}, 0 \right\rangle = \left\langle 0, 0, \frac{-2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} \right\rangle \neq \vec{0}.$$

So, $\left\langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right\rangle$ is not conservative. Thus, $\int_C \frac{x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy$ is not path independent.

(d) If $\oint_C \vec{F}\cdot d\vec{r} = 0$ for a simple closed curve C, then $\vec{F}$ is conservative.

Consider $\vec{F} = \langle 0, 0, 1 - x^2 - y^2 \rangle$. Then curl$\vec{F} = \langle -2y, 2x, 0 \rangle \neq \vec{0}$, so $\vec{F}$ is not conservative. However, for $C : \vec{r}(t) = \langle \cos\theta, \sin\theta, 0 \rangle$, $0 \le t \le 2\pi$, a simple closed curve,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\left(\vec{r}(t)\right)\cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \vec{0}\cdot \vec{r}\,'(t)\,dt = 0.$$

(e) If $\nabla^2 f = \nabla\cdot\nabla f \equiv 0$ then $\int_C f_y\,dx - f_x\,dy$ is path independent.
Let C be a simple closed curve.

$$\begin{aligned}\int_C f_y\,dx - f_x\,dy &\underset{\text{Green's}}{=} \iint_R \frac{\partial}{\partial x}(-f_x) - \frac{\partial}{\partial y}(f_y)\,dA \\ &= -\iint_R f_{xx} + f_{yy}\,dA \\ &= -\iint_R \nabla^2 f\,dA \\ &= -\iint_R 0\,dA \\ &= 0.\end{aligned}$$

(f) There is a vector field $\vec{F}$ such that curl$\vec{F} = \langle 2x, 3yz, -xz^2 \rangle$.
Recall that for any vector field $\vec{F}$, divcurl$\vec{F} = 0$. Since div $\langle 2x, 3yz, -xz^2 \rangle = 2 + 3z + x \neq 0$, then $\langle 2x, 3yz, -xz^2 \rangle$ is not the curl of any vector field.