Calculus I Final Review Solutions

1. In general an exponential function through (x_1, y_1), (x_2, y_2) with horizontal asymptote $y = a$ is $y = a + (y_1 - a)\left(\frac{y_2 - a}{y_1 - a}\right)^{\frac{x - x_1}{x_2 - x_1}}$. (Verify this!) So, we have the equations $y = 5\left(\frac{12}{5}\right)^{\frac{x-1}{2}}$ and $y = 20 - 15\left(\frac{8}{15}\right)^{\frac{x-1}{2}}$. Alternatively, you can find specific a and b for the equations $y = ab^x$ and $y = 20 + ab^x$ by preforming algebra. So, for example in the second case, we have

$$5 = 20 + ab \qquad 12 = 20 + ab^3$$
$$-15 = ab \qquad -8 = ab \cdot b^2$$
$$-8 = -15b^2$$
$$\left(\frac{8}{15}\right)^{\frac{1}{2}} = b$$
$$a = -15b^{-1} = -15\left(\frac{8}{15}\right)^{\frac{-1}{2}}$$
$$y = 20 + ab^x = 20 - 15\left(\frac{8}{15}\right)^{\frac{-1}{2}}\left(\frac{8}{15}\right)^{\frac{x}{2}}$$
$$y = 20 - 15\left(\frac{8}{15}\right)^{\frac{x-1}{2}}$$

2.

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$$
$$.1 = \frac{A(t)}{A_0} = \left(\frac{1}{2}\right)^{\frac{t}{125}}$$
$$\ln(.1) = \frac{-t}{125}\ln 2$$
$$t = \frac{125\ln(10)}{\ln 2} \approx 415.24 \text{ days}$$

3. Two examples are $y = 3 + 5\sin\left(\frac{\pi}{6}(x+1)\right) = 3 + 5\cos\left(\frac{\pi}{6}(x-2)\right)$.

4. First, each piece is a polynomial, so the only place f may not be continuous or differentiable is at $x = c$. So, we need to check if the function values and derivatives agree on either side of c.

$$c^3 - 2c^2 + 3c - 2 = 2c - 2$$
$$0 = c^3 - 2c^2 + c$$
$$= c(c^2 - 2c + 1)$$
$$= c(c-1)^2$$

So, f is continuous if $c = 0$ or $c = 1$. Now, do the derivatives agree over these c's?

$$c = 0: \ 3(0)^2 - 4(0) + 3 = 3 \neq 2 \qquad c = 1: \ 3(1)^2 - 4(1) + 3 = 2 \overset{\checkmark}{=} 2$$

So, f is differentiable if $c = 1$.

5. (a) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{8x^2+1} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{1/x}{16x} = \lim_{x\to\infty}\frac{1}{16x^2} = 0.$

   (b) $\displaystyle\lim_{x\to 0^+}(\sin x)^{\ln x} = \text{“}0^{-\infty}\text{”} = \text{“}\frac{1}{0^{\infty}}\text{”} = \text{“}\frac{1}{0}\text{”} = \infty$

(c) $\lim_{x\to 1}(\ln x)^{\ln x} = \lim_{x\to 1} e^{\ln(x)\ln(\ln x)} = e^{\lim_{x\to 1}\frac{\ln(\ln x)}{1/\ln(x)}} \overset{\text{L'H}}{=} e^{\lim_{x\to 1}\frac{-1/\ln(x)\cdot 1/x}{-1/(\ln x)^2\cdot 1/x}} = e^{\lim_{x\to 1}\ln(x)} = e^0 =$ 1.

(d) $\lim_{x\to \ln 2}\dfrac{\sinh x - \frac{3}{4}}{x-\ln 2} = \dfrac{d}{dx}[\sinh x]\Big|_{x=\ln 2} = \cosh\ln 2 = \dfrac{2+1/2}{2} = \dfrac{3}{4}.$

(e) $\lim_{x\to\infty} x - \sqrt{x^2+x}\cdot\dfrac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\dfrac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\dfrac{-x}{x+\sqrt{x^2+x}}\cdot\dfrac{1/x}{1/x} =$ $\lim_{x\to\infty}\dfrac{-1}{1+\sqrt{1+\frac{1}{x}}} = \dfrac{-1}{1+\sqrt{1+0}} = -\dfrac{1}{2}.$

(f) $\lim_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ Does not exist. Consider $x_n = \frac{\pi}{2}+2\pi n$ and $y_n = \frac{3\pi}{2}+2\pi n$.

(g)

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$
$$0 = \lim_{x\to 0} -|x| \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0}|x| = 0$$
$$\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$

6. (a) $v'(2) \approx \dfrac{v(3)-v(1)}{3-1} = \dfrac{-10}{2} = -5 \text{ ft/sec}^2.$

(b) $\dfrac{v(4)-v(0)}{4-0} = \dfrac{-20}{4} = -5 \text{ ft/sec}^2.$

(c) $1\cdot(32+26+22+20+19) = 119\text{ft} \le D \le 1\cdot(40+32+26+22+20) = 140\text{ft}.$

(d) Using trapezoid approximation of $s(t) = 15 + \int_0^t v(u)\,du$:

t (sec)	0	1	2	3	4	5
$s(t)$ (ft)	15	51	80	104	125	144.5

7.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x^2+1}\right] &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2+1}-\frac{1}{x^2+1}}{h}\\ &= \lim_{h\to 0}\frac{x^2+1-((x+h)^2+1)}{h((x+h)^2+1)(x^2+1)}\\ &= \lim_{h\to 0}\frac{-2xh-h^2}{h((x+h)^2+1)(x^2+1)}\\ &= \lim_{h\to 0}\frac{-2x-h}{((x+h)^2+1)(x^2+1)}\\ &= \frac{-2x}{(x^2+1)^2}.\end{aligned}$$

8. (a) After growing for 40 days, the volume of the yam is 22 cm^3.

(b) During its 35th day of growth, the volume of the yam should increase by approximately .8 cm^3.

(c) When the yam has volume 25 cm^3, it should have been growing for 60 days.

(d) In order for the yam to increase its volume from 22 cm^3 to 23 cm^3, it should grow for approximately 1.25 days.

(e) The yam gained 11 cm^3 of volume between day 10 and day 20.

(f) Recall the equation relating derivatives and inverses, $\left(V^{-1}\right)'(y) = \dfrac{1}{V'(x)}$ where $V(x) = y$. So, $V'(40) = \dfrac{1}{(V^{-1})'(V(40))} = \dfrac{1}{(V^{-1})'(22)} = \dfrac{1}{1.25} = .8\ \text{cm}^3/\text{day}$.

9. The tangent line at $x = 0$ is $y = f'(0)x + f(0) = -2x + 6$. Since f is concave down, we have that $f(x) < -2x + 6$ for $x \neq 0$. Note that f is continuous and $f(3) < -2(3) + 6 = 0 < 6 = f(0)$. So, by the Intermediate Value Theorem, $f(c) = 0$ for some $0 < c < 3$. Also, since f is continuous and decreasing, for $a < c < b$ we have that $f(a) < f(c) = 0 < f(b)$. So, $f(x) \neq 0$ for $x \neq c$. Thus, $f(x)$ has only one zero, at $x = c$. Finally, $f(-2) < -2(-2) + 6 = 10$. So, $f(-2) \neq 12$ and $f(-2) \neq 10$. Also, since f is decreasing, we have $f(-2) > f(0) = 6$. So, $f(-2) \neq 4$. Thus, $f(-2)$ can be any value such that $6 < f(-2) < 10$.

10.

$$
\begin{aligned}
y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 2x \\
4 + 12\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{1}{3}.
\end{aligned}
$$

So, the tangent line is $y = -\frac{1}{3}x + 2$.

11.

$$
\begin{aligned}
\sinh x \cosh y + \cosh x \sinh x &= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}\right) \\
&= \frac{1}{4}\left(2e^{x+y} - 2e^{-x-y}\right) \\
&= \frac{e^{x+y} - e^{-(x+y)}}{2} \\
&= \sinh(x + y).
\end{aligned}
$$

12. Using the tangent line approximation, we have $\ln x \approx \dfrac{1}{a}(x - a) + \ln a$ for $x \approx a$. In particular, $\ln(1.02) \approx \frac{1}{1}(1.02 - 1) + \ln(1) = .02$. Since $\ln x$ is concave down, this is an over-approximation.

13.

$$
\begin{aligned}
f'(x) &= 12x^3 - 12x^2 = 12x^2(x - 1) \\
f''(x) &= 36x^2 - 24x = 36x\left(x - \frac{2}{3}\right)
\end{aligned}
$$

$f(-2) = 22$ $\qquad$ $f(2) = 36$ Global Max

$f(0) = 6$ $\qquad$ $f(1) = 5$ Global Min

Using first and second derivative sign lines, we have that f is

Increasing: $(1, \infty)$ $\qquad$ Decreasing: $(-\infty, 0) \cup (0, 1)$

Concave Up: $(-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$ $\qquad$ Concave Down: $\left(0, \frac{2}{3}\right)$

14. First, note the maximum/minimum of a positive function occurs at the same point as the square of the function. So, we will be minimizing the square of the distance to simplify computation.

$$D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2 = x^4 - 4x + \frac{17}{4}$$

$$\left(D^2\right)' = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2+x+1) = 0.$$

So, we have only one critical point, $x = 1$, which by looking at a sign line, is a local min. So, the global min occurs at $x = 1$. This corresponds to the point (1,1).

15. $V = 100\pi = \pi r^2 h \Rightarrow h = \dfrac{100}{r^2}$ and $r = \sqrt{\dfrac{100}{h}}$. So, at the time we are interested in, we have $r = \sqrt{\dfrac{100}{25}} = \sqrt{4} = 2$, and so, we have $h' = -\dfrac{200}{r^3} r' = -\dfrac{200}{8}(-3) = 75$ cm/min.

16. (a) $(x'(t), y'(t)) = (e^t, 10e^{2t}) \Rightarrow (x'(\ln 3), y'(\ln 3)) = (e^{\ln 3}, 10e^{\ln 9}) = (3, 90)$. Also, $(x(\ln 3), y(\ln 3)) = (3, 45)$. So, $(x_\ell, y_\ell) = (3(t - \ln 3) + 3, 90(t - \ln 3) + 45)$.

    (b) Eliminating the parameter, we get:

$$\frac{x_\ell - 3}{3} = t - \ln 3 = \frac{y_\ell - 45}{90}$$
$$y_\ell = 30(x_\ell - 3) + 45$$

    (c) $\sqrt{(x'(\ln 3))^2 + (y'(\ln 3))^2} = \sqrt{e^{2t} + 100e^{4t}}\Big|_{t=\ln 3} = \sqrt{9 + 8100} = 3\sqrt{901}$.

    (d) All three parametric equations satisfy $y = 5x^2$, However, the first two do not have the same x-domain as the original function, and so, they have different graphs. The third has the same domain, and so, has the same graph.

17. $\displaystyle\int_0^1 x^2 - x^3\,dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

18. $\displaystyle\int_0^5 f(x-3)\,dx = \int_{-3}^2 f(x)\,dx = \int_{-3}^3 F(x)\,dx - \int_2^3 f(x)\,dx = 2\int_{-3}^0 f(x)\,dx - 5 = 2(2) - 5 = -1$. So, $\displaystyle\int_0^5 f(x-3) - 2\,dx = -1 - 2(5) = -11$.

19. $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. Consider $f(x) = \sin x$ on $\left[\frac{\pi}{2}, 2\pi\right]$. Notice that the left integral adds positive and (possibly) negative areas, then makes the resulting number positive. The right integral makes all areas positive, then adds them up. Since in the second case there can be no cancelation, it has to be larger. The only case for the two to be equal is if f is never negative (or, more precisely, nonnegative almost everywhere).

20. $\displaystyle 1 = \frac{1}{2c - c}\int_c^{2c} \frac{1}{x}\,dx = \frac{1}{c}\left[\ln x\right]_c^{2c} = \frac{1}{c}\left(\ln(2c) - \ln(c)\right) = \frac{1}{c}\ln\left(\frac{2c}{c}\right) = \frac{1}{c}\ln 2$. So, $c = \ln 2$.

21.

$$s(t) = \frac{1}{2}at^2 + 100$$
$$0 = s(5) = \frac{1}{2}a(25) + 100$$
$$a = \frac{-100 \cdot 2}{25} = -8.$$

So, the acceleration due to gravity is 8 m/s^2.

22.

$$\begin{aligned}
y &= e^x - 4\cos x + C \\
2 &= 1 - 4 + C \\
5 &= C \\
y &= e^x - 4\cos x + 5.
\end{aligned}$$

Alternatively,

$$\begin{aligned}
y(x) &= y(0) + \int_0^x (e^t + 4\sin t)\,dt \\
&= 2 + \left[e^t - 4\cos t\right]_0^x \\
&= 2 + (e^x - 4\cos x) - (e^0 - 4\cos 0) \\
&= 5 + e^x - 4\cos x
\end{aligned}$$

23. First, since there is no area underneath a point, we have

$$F(\sqrt{2}) = \int_2^2 \frac{1}{t^2+1}\,dt = 0.$$

Now, by the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx}\int_2^{x^2} \frac{1}{t^2+1}\,dt = \frac{1}{(x^2)^2+1}\cdot\frac{d}{dx}x^2 = \frac{2x}{x^4+1}.$$

So,

$$F'(\sqrt{2}) = \frac{2\sqrt{2}}{5}.$$

Finally,

$$F''(x) = \frac{2(x^4+1) - 4x^3(2x)}{(x^4+1)^2} = \frac{-6x^4+2}{(x^4+1)^2}.$$

So

$$F''(\sqrt{2}) = -\frac{22}{25}.$$