Derivative Challenge Problems

1. (a) Sketch a graph of a continuous function f on [0,1] with range contained in [0,1]. Does f have a fixed point? If so, find all such points.

   (b) Try to draw a graph of a continuous function with domain [0,1], range contained in [0,1], and with no fixed point. What is the problem?

   (c) Show that every such function must have a fixed point.

2. Suppose $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all x and y and

$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$

   Find $f(0)$, $f'(0)$, $f'(x)$, $f(x)$.

3. Suppose f and g are defined in an interval around x_0, , f is differentiable at x_0, $f(x_0) = 0$, and g is continuous at x_0. Show that fg is differentiable at x_0.

4. The Binomial Theorem, from algebra, states that

$$(a+b)^n = a^n + na^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + nab^{n-1} + b^n,$$

   where $\binom{n}{k} = \dfrac{n(n+1)(n+2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(1)}$.
   Show that for $n-$differentiable functions f, g, we have

$$\frac{d^n}{dx^n}(fg)(x) = \frac{d^nf}{dx^n} + n\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \cdots + \binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^kg}{dx^k} + \cdots + \frac{d^ng}{dx^n}.$$