

1

$$\begin{aligned} & \exists v_2 \exists v_3 [v_2 < v_3 \wedge v_3 < v_0 \wedge \forall v_4 [v_4 < v_0 \rightarrow v_4 = v_2 \vee v_4 = v_3]] \\ & \wedge \exists v_5 [v_0 < v_5 \wedge v_5 < v_1 \wedge \forall v_6 [v_0 < v_6 \wedge v_6 < v_1 \rightarrow v_6 = v_5]] \end{aligned}$$

2

$$\begin{aligned} & \{v_0 < v_1\} \vdash v_0 < v_1 \\ & \{v_0 < v_1\} \vdash \forall v_1 (v_0 < v_1) \quad \text{gen.} \\ & \{v_0 < v_1\} \vdash \forall v_1 (v_0 < v_1) \rightarrow v_0 < v_0 \quad \text{by 3.27} \\ & \{v_0 < v_1\} \vdash v_0 < v_0 \quad \text{MP} \end{aligned}$$

3

$$\vdash \exists v_0 [v_0 < v_1 \wedge v_2 = v_3] \vee \exists v_0 [v_0 < v_2] \leftrightarrow \exists v_0 [(v_0 < v_1 \wedge v_2 < v_3) \vee v_0 < v_2] \quad 3.45$$

4 Suppose that  $\Gamma$  is consistent, and by the completeness theorem let  $\bar{A}$  be a model of  $\Gamma$ . Take any  $a \in A$ , and by (4) applied twice, get  $b, c \in A$  such that  $a < b$  and  $b < c$ . Note that  $a < c$  by (2). By (4) we have  $a = b$  or  $a = c$  or  $b = c$ . If  $a = b$ , then (1) is contradicted. If  $b = c$ , again (1) is contradicted. If  $a = c$ , then again (1) is contradicted.

5 Add an individual constant to the (implicit) language  $\mathcal{L}$ , obtaining a language  $\mathcal{L}'$ . Let  $\Gamma'$  consist of  $\Gamma$  together with the following sentences for each  $n \geq 2$ :

$$\exists v_0 \dots \exists v_{n-1} \left( \bigwedge_{i < n} (v_i < \mathbf{c}) \wedge \bigwedge_{i < j < n} [\neg(v_i = v_j)] \right)$$

We claim that every finite subset  $\Delta$  of  $\Gamma'$  has a model. For, in  $\Delta$  there are only finitely many of the above sentences, say those associated with  $n_0, \dots, n_k$ . Let  $m$  be the maximum of  $n_0, \dots, n_k$ , and define the structure  $\bar{A}$  to be  $(\omega, <)$  together with  $\mathbf{c}^{\bar{A}} = m + 1$ . Clearly  $\bar{A}$  is a model of  $\Delta$ .

So by the compactness theorem  $\Gamma'$  has a model  $\bar{B}$ . Then for each positive integer  $m$  there are at least  $m$  elements less than  $\mathbf{c}^{\bar{B}}$ . The reduct of  $\bar{B}$  to the language  $\mathcal{L}$  is as desired.