Solutions for review

$\boxed{1}$

$$\begin{array}{ll}
(1) & \{\neg(v_0 = v_1)\} \vdash \neg(v_0 = v_1) \\
(2) & \{\neg(v_0 = v_1)\} \vdash \forall v_1(\neg(v_0 = v_1)) \quad \text{(generalization)} \\
(3) & \{\neg(v_0 = v_1)\} \vdash \forall v_1(\neg(v_0 = v_1)) \rightarrow \neg(v_0 = v_0) \quad \text{(Theorem 3.27)} \\
 & \{\neg(v_0 = v_1)\} \vdash \neg(v_0 = v_0) \quad \text{(2), (3), MP}
\end{array}$$

Thus $\{\neg(v_0 = v_1)\}$ is inconsistent, by Lemma 4.1.

Alternate proof: suppose that $\{\neg(v_0 = v_1)\}$ is consistent. Since $\{\neg(v_0 = v_1)\} \vdash \forall v_0 \forall v_1(\neg(v_0 = v_1))$ by generalization twice, also $\{\forall v_0 \forall v_1(\neg(v_0 = v_1))\}$ is consistent. By the compactness theorem, let $\overline{A}$ be a model of $\{\forall v_0 \forall v_1(\neg(v_0 = v_1))\}$. Then for all $a, b \in A$ we have $a \neq b$. This is a contradiction, since $a = a$ for any $a \in A$.

$\boxed{2}$ Let φ be $\exists v_0(\neg(v_0 = v_1))$ and let v_j be v_1. Then $\mathrm{Subb}^{v_0}_{v_j}\varphi$ is $\exists v_1(\neg(v_1 = v_1))$. Now in a structure $\overline{A}$ which has at least two elements, for any assignment $a : \omega \rightarrow A$ we have $\overline{A} \models \varphi[a]$ but $\overline{A} \not\models \mathrm{Subb}^{v_0}_{v_j}\varphi[a]$. Hence $\overline{A} \not\models (\varphi \leftrightarrow \mathrm{Subb}^{v_0}_{v_1}\varphi)[a]$. By Theorem 3.2, $\overline{A} \not\vdash (\varphi \leftrightarrow \mathrm{Subb}^{v_0}_{v_1}\varphi)$.

$\boxed{3}$

$$\begin{array}{ll}
(1) & \vdash (\varphi \rightarrow \exists v_i\psi) \leftrightarrow (\neg\varphi \vee \exists v_i\psi) \\
(2) & \vdash \varphi \leftrightarrow \forall v_i\varphi \quad 3.29 \\
(3) & \vdash \varphi \wedge \forall v_i\neg\psi \leftrightarrow \forall v_i\varphi \wedge \forall v_i\neg\psi \quad (2),\ 3.20 \\
(4) & \vdash (\forall v_i\varphi \wedge \forall v_i\neg\psi) \leftrightarrow \forall v_i(\varphi \wedge \neg\psi) \quad 3.41 \\
(5) & \vdash \varphi \wedge \neg\psi \leftrightarrow \neg(\varphi \rightarrow \psi) \quad \text{taut.} \\
(6) & \vdash \forall v_i(\varphi \wedge \neg\psi) \leftrightarrow \forall v_i(\neg(\varphi \rightarrow \psi)) \quad (5),\ 3.43 \\
(7) & \vdash \varphi \wedge \forall v_i\neg\psi \leftrightarrow \forall v_i(\neg(\varphi \rightarrow \psi)) \quad (3), (4), (6), \text{taut.} \\
(8) & \vdash \neg\varphi \vee \exists v_i\psi \leftrightarrow \exists v_i(\varphi \rightarrow \psi) \quad (7), \text{taut.} \\
 & \vdash (\varphi \rightarrow \exists v_i\psi) \leftrightarrow \exists v_i(\varphi \rightarrow \psi) \quad (1), (8), \text{taut.}
\end{array}$$

$\boxed{4}$ By a change of bound variables the indicated formula is equivalent to $\forall v_0(v_0 < v_1 \rightarrow \exists v_2(v_2 < v_0))$. Hence a formula in prenex normal form equivalent to the formula is $\forall v_0 \exists v_2(v_0 < v_1 \rightarrow v_2 < v_0)$,

In more detail:

$$\begin{array}{l}
\vdash \exists v_1(v_1 < v_0) \leftrightarrow \exists v_2(v_2 < v_0) \quad 3.25 \\
\vdash [v_0 < v_1 \rightarrow \exists v_1(v_1 < v_0)] \leftrightarrow [v_0 < v_1 \rightarrow \exists v_2(v_2 < v_0)] \quad 3.20 \\
\vdash [v_0 < v_1 \rightarrow \exists v_2(v_2 < v_0)] \leftrightarrow \exists v_2[v_0 < v_1 \rightarrow v_2 < v_0] \quad \text{number 3} \\
\vdash \forall v_0[v_0 < v_1 \rightarrow \exists v_2(v_2 < v_0)] \leftrightarrow \forall v_0 \exists v_2[v_0 < v_1 \rightarrow v_2 < v_0] \quad 3.43
\end{array}$$

$\boxed{5}$ (a): both occurrences of v_0 are free. Both occurrences of v_1 are free. Both occurrences of v_2 are bound. The occurrence of v_3 is free.

(b): $\mathrm{Subf}^{v_0}_{v_1+v_2}\varphi$ is $v_1 + v_2 = v_1 \wedge \forall v_2(v_2 = v_1 + v_2 \rightarrow v_1 = v_3)$.

(c): $\mathrm{Subb}^{v_2}_{v_1}\varphi$ is $v_0 = v_1 \wedge \forall v_1(v_1 = v_0 \rightarrow v_1 = v_3)$.

(d): $\mathrm{Subb}^{v_2}_{v_1}\mathrm{Subf}^{v_0}_{v_1+v_2}\varphi$ is $v_1 + v_2 = v_1 \wedge \forall v_1(v_1 = v_1 + v_2 \rightarrow v_1 = v_3)$.

(e): $\mathrm{Subf}^{v_0}_{v_1+v_2}\mathrm{Subb}^{v_2}_{v_1}\varphi$ is $v_1 + v_2 = v_1 \wedge \forall v_1(v_1 = v_1 + v_2 \rightarrow v_1 = v_3)$.

$\boxed{6}$ $v_0 < v_1 \wedge v_1 < v_2 \wedge \exists v_3(v_0 < v_3 \wedge v_3 < v_1) \wedge \neg\exists v_3(v_1 < v_3 \wedge v_3 < v_2)$.

$\boxed{7}$ Let $\overline{A} = (A, R)$ be any structure for the indicated language, and let $a : \omega \rightarrow A$ be any assignment. Assume that $\overline{A} \models \exists v_0[v_0 = v_1 \wedge \exists v_1(v_0 < v_1)][a]$. Choose $x \in A$ so that $\overline{A} \models [v_0 = v_1 \wedge \exists v_1(v_0 < v_1)][a^0_x]$. Thus $x = a_1$. Choose $y \in A$ so that $\overline{A} \models (v_0 < v_1)[a^{0\,1}_{x\,y}$. Thus $(x, y) \in \mathbf{R}^{\overline{A}}$.

Now to show that $\overline{A} \models \forall v_0[v_0 = v_1 \rightarrow \exists v_1(v_0 < v_1)][a]$, let $z \in A$ be arbitrary and assume that $\overline{A} \models (v_0 = v_1)[a^0_z]$. Thus $z = a_1 = x$. So $(z, y) \in \mathbf{R}^{\overline{A}}$, and it follows that $\overline{A} \models \exists v_1(v_0 < v_1)[a^0_z]$, as desired.