Solutions for review questions for the final

$\boxed{1}$

$$\begin{array}{cccccccccccccccccc}
 & 11 & & 12 & 6 & 10 & 9 & 8 & 5 & 6 & 2 & 4 & 1 & 8 & 3 & 7 & 2 & 3 \\
([\{[S_1 & \to & (\neg\neg S_2 & \to & S_2))] \to & \neg & \neg & S_0] & \to & S_3] & \to & S_4) & \to & [(S_0 & \to & S_3) & \to & S_4] \\
 & 0 & & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}$$

The implication on the far left is inconsistently assigned, so the formula is a tautology.

$\boxed{2}$ A formula built up from $\neg$ and sentential variables must have the form $\neg\cdots\neg S_i$ for some $i$. Such a formula is either equivalent to $S_i$ (for an even number of $\neg$'s), or to $\neg S_i$ (for an odd number). Suppose that $S_0 \to S_0$ is equivalent to $\neg\cdots\neg S_i$.

*Case 1.* An even number of $\neg$'s before $S_i$. Assigning 0 to all $S_j$'s gives the value 1 to $S_0 \to S_0$ and 0 to $\neg\cdots\neg S_i$, contradiction.

*Case 2.* An odd number of $\neg$'s before $S_i$. Assigning 1 to all $S_j$'s gives the value 1 to $S_0 \to S_0$ and 0 to $\neg\cdots\neg S_i$, contradiction.

$\boxed{3}$ We need several lemmas.

**Lemma 1.** *For any formula $\varphi$, $\vdash \neg\neg\varphi \to \varphi$.*

**Proof.**

$$\begin{aligned}
\{\neg\neg\varphi\} &\vdash \neg\neg\neg\neg\varphi \to \neg\neg\varphi && \text{using axiom (1)}\\
\{\neg\neg\varphi\} &\vdash \neg\varphi \to \neg\neg\neg\varphi && \text{using axiom (3)}\\
\{\neg\neg\varphi\} &\vdash \neg\neg\varphi \to \varphi && \text{using axiom (3)}\\
\{\neg\neg\varphi\} &\vdash \varphi &&\\
&\vdash \neg\neg\varphi \to \varphi && \square
\end{aligned}$$

**Lemma 2.** *For any formula $\varphi$, $\vdash \varphi \to \neg\neg\varphi$.*

**Proof.**

$$\begin{aligned}
\{\varphi, \neg\neg\neg\varphi\} &\vdash \neg\varphi && \text{using Lemma 1}\\
\{\varphi\} &\vdash \neg\neg\neg\varphi \to \neg\varphi &&\\
\{\varphi\} &\vdash \varphi \to \neg\neg\varphi && \text{using axiom (3)}\\
\{\varphi\} &\vdash \neg\neg\varphi &&\\
&\vdash \varphi \to \neg\neg\varphi &&
\end{aligned}$$

**Lemma 3.** *For any formulas $\varphi, \psi$, $\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$.*

$$\begin{aligned}
\{\varphi \to \neg\psi, \psi\} &\vdash \neg\neg\varphi \to \varphi && \text{using Lemma 1}\\
\{\varphi \to \neg\psi, \psi\} &\vdash \neg\neg\varphi \to \neg\psi && \text{using 1.14}\\
\{\varphi \to \neg\psi, \psi\} &\vdash \psi \to \neg\varphi && \text{using axiom (3)}\\
\{\varphi \to \neg\psi, \psi\} &\vdash \neg\varphi &&\\
\{\varphi \to \neg\psi\} &\vdash \psi \to \neg\varphi &&\\
&\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi) &&
\end{aligned}$$

**Lemma 4.** *For any formulas $\varphi, \psi$, $\vdash \varphi \to (\psi \to (\varphi \wedge \psi))$.*

**Proof.**

$$\begin{aligned}
\{\varphi, \psi, \varphi \to \neg\psi\} &\vdash \psi \\
\{\varphi, \psi, \varphi \to \neg\psi\} &\vdash \neg\psi \\
\{\varphi, \psi, \varphi \to \neg\psi\} &\vdash \neg(\varphi \to \varphi) && \text{using 1.13} \\
\{\varphi, \psi\} &\vdash (\varphi \to \neg\psi) \to \neg(\varphi \to \varphi) \\
\{\varphi, \psi\} &\vdash (\varphi \to \varphi) \to \neg(\varphi \to \neg\psi) && \text{using Lemma 3} \\
\{\varphi, \psi\} &\vdash \neg(\varphi \to \neg\psi) && \text{using 1.11} \\
&\vdash \varphi \to (\psi \to (\varphi \wedge \psi))
\end{aligned}$$

Now the exercise follows from Lemmas 1, 2, and 4.

$\boxed{4}$

$$\neg S_0 \vee \neg S_1 \vee (\neg S_2 \wedge \neg S_3) \vee (S_2 \wedge S_3)$$

$\boxed{5}$ Suppose that the given expression is a term. Then by Proposition 2.2(ii)(c) there are terms $\sigma, \tau$ such that

$$\langle +, v_0, +, v_0, v_1, v_2 \rangle = \langle + \rangle^\frown \sigma^\frown \tau$$

Hence

$$\langle v_0, +, v_0, v_1, v_2 \rangle = \sigma^\frown \tau.$$

Now $v_0$ is a term, and it is a segment of $\sigma$. So by Proposition 2.2(iii) we have $v_0 = \sigma$. Hence

$$\langle +, v_0, v_1, v_2 \rangle = \tau$$

But then the term $\langle +, v_0, v_1 \rangle$ is a proper initial segment of $\tau$, contradicting Proposition 2.2(iii).

$\boxed{6}$ Suppose that the given expression is a formula. Then by Proposition 2.6(ii)(e) there is a formula $\varphi$ such that

$$\langle \forall, v_0, =, v_1, v_0, v_2, v_3 \rangle = \langle \forall, v_0 \rangle^\frown \varphi \rangle.$$

It follows that

$$\langle =, v_1, v_0, v_2, v_3 \rangle = \varphi.$$

But $\langle =, v_1, v_0 \rangle$ is a formula which is a proper initial segment of $\varphi$, contradicting Proposition 2.6(iii).

$\boxed{7}$

$$\neg(v_0 = 0) \wedge \neg(v_0 = S0) \wedge \forall v_1[\exists v_2(v_1 \cdot v_2 = v_0) \to v_1 = S0 \vee v_1 = v_0]$$

$\boxed{8}$

$$\forall v_0[0 < v_0 \to \exists v_1(v_1 \cdot v_1 = v_0)]$$

$\boxed{9}$ Below each variable we write $f$ for free occurrence and $b$ for bound occurrence.

$$\begin{array}{cccccccccccccccccccccccccc}
\forall & v_0 & ( & v_0 & + & v_1 & = & v_2 & \vee & \exists & v_1 & ( & v_1 & + & v_0 & = & v_2 & \wedge & \forall & v_2 & ( & v_3 & = & v_2 & ) & ) \\
 & b & & b & & f & & f & & & b & & b & & b & & f & & & b & & f & & b & &
\end{array}$$

$\boxed{10}$

$$\forall v_0 \exists v_2 \exists v_4 [v_0 > 0 \to (v_1 = v_3 \to v_0 = v_1)]$$

$\boxed{11}$ It suffices to find a model of the indicated set. Let $\overline{M} = (\{0,1\}, f)$, where $f(0) = 1$ and $f(1) = 0$.

$\boxed{12}$ By 3.37 we have $\vdash \exists v_1 \forall v_2 \varphi \to \forall v_2 \exists v_1 \varphi$. Hence by generalization on $v_0$ and axiom (L2) we get $\vdash \forall v_0 \exists v_1 \forall v_2 \varphi \to \forall v_0 \forall v_2 \exists v_1 \varphi$.

$\boxed{13}$ In the language for $(\omega, S, 0, +, \cdot)$ let $\varphi$ be the formula $\forall v_0(\neg(Sv_0 = v_1))$. Then $\mathrm{Subf}^{v_1}_0 \varphi$ is $\forall v_0(\neg(Sv_0 = 0))$. and $P \vdash \mathrm{Subf}^{v_1}_0 \varphi$. But $P \not\vdash \varphi$. In fact, let $a : \omega \to \omega$ be such that $a_1 = 1$. Then $\mathrm{not}((\omega, S, 0, +, \cdot) \models \varphi[a])$, hence $\mathrm{not}(P \models \varphi)$, hence $\mathrm{not}(P \vdash \varphi)$.

$\boxed{14}$ We assume that $x$ is $v_0$ and $y$ is $v_1$. Let $\overline{M}$ be any $\mathscr{L}$-structure, and assume that $\overline{M} \models \exists v_0 F v_0 \wedge \forall v_0 \forall v_1 [F v_0 \wedge F v_1 \to v_0 = v_1]$. This means that there is a $u \in M$ such that $F^{\overline{M}} = \{u\}$.

Now suppose that $a : \omega \to M$ is any assignment. First suppose that $\overline{M} \models \exists v_0 [F v_0 \wedge G v_0 v_1][a]$. This means that $(u, a_1) \in G^{\overline{M}}$. Now take any $w \in M$. If $\overline{M} \models F v_0[a^0_w]$, then $w = u$ and so $(w, a_1) \in R^{\overline{M}}$, and consequently $\overline{M} \models G v_0 v_1 [a^0_w]$. Thus $\overline{M} \models (F v_0 \to G v_0 v_1)[a^0_w]$. Since $w$ is arbitrary, this proves that $\overline{M} \models \forall v_0 (F v_0 \to G v_0 v_1)$.

Second suppose that $\overline{M} \models \forall v_0 (F v_0 \to G v_0 v_1)[a]$. By definition of modeling of $\forall$ it follows that $\overline{M} \models (F v_0 \to G v_0 v_1)[a^0_u]$. Since $F^{\overline{M}} = \{u\}$, it follows that $\overline{M} \models G v_0 v_1 [a^0_u]$. So $\overline{M} \models (F v_0 \wedge G v_0 v_1)[a^0_u]$, so $\overline{M} \models \exists v_0 (F v_0 \wedge G v_0 v_1)$.

$\boxed{15}$ We go through the proofs of 4.20–4.24 for this formula.

(1) $\vdash v_0 + v_1 \cdot v_1 = 0 \leftrightarrow \exists v_2 [v_0 + v_1 \cdot v_1 = v_2 \wedge 0 = v_2]$ (1) in proof of 4.24

(2) $\vdash 0 = v_2 \leftrightarrow \exists v_0 [v_0 = v_2 \wedge 0 = v_0]$ 4.20

(3) $\vdash v_0 + v_1 \cdot v_1 = v_2 \leftrightarrow \exists v_3 \exists v_4 \exists v_5 [v_0 = v_3 \wedge v_1 \cdot v_1 = v_4 \wedge v_2 = v_5 \wedge v_3 + v_4 = v_5]$
(∗) in proof of 4.23 with $n = 3$

(4) $\vdash v_1 \cdot v_1 = v_4 \leftrightarrow \exists v_5 \exists v_6 \exists v_7 [v_1 = v_5 \wedge v_1 = v_6 \wedge v_4 = v_7 \wedge v_5 \cdot v_6 = v_7]$
(∗) in proof of 4.23 with $n = 5$

(5) $\vdash v_5 \cdot v_6 = v_7 \leftrightarrow \exists v_0 \exists v_1 \exists v_2 [v_0 = v_5 \wedge v_1 = v_6 \wedge v_2 = v_7 \wedge v_0 \cdot v_1 = v_2]$
4.22

(6) $\vdash v_3 + v_4 = v_5 \leftrightarrow \exists v_0 \exists v_1 \exists v_2 [v_0 = v_3 \wedge v_1 = v_4 \wedge v_2 = v_5 \wedge v_0 + v_1 = v_2]$
4.22

(7) $\vdash v_2 + v_3 = S0 \leftrightarrow \exists v_4 [v_2 + v_3 = v_4 \wedge S0 = v_4]$ (1) in proof of 4.24

(8) $\vdash v_2 + v_3 = v_4 \leftrightarrow \exists v_5 \exists v_6 \exists v_7 [v_2 = v_5 \wedge v_3 = v_6 \wedge v_4 = v_7 \wedge v_5 + v_6 = v_7]$
(∗) in proof of 4.23 with $n = 5$

(9) $\vdash v_5 + v_6 = v_7 \leftrightarrow \exists v_0 \exists v_1 \exists v_2 [v_5 = v_0 \wedge v_6 = v_1 \wedge v_7 = v_2 \wedge v_0 + v_1 = v_2]$
4.22

(10) $\vdash S0 = v_4 \leftrightarrow \exists v_5 \exists v_6 [0 = v_5 \wedge v_4 = v_6 \wedge S v_5 = v_6]$
(∗) in proof of 4.23 with $n = 5$

(11) $\vdash 0 = v_5 \leftrightarrow \exists v_0[v_0 = v_5 \wedge 0 = v_0]$ 4.20

(12) $\vdash Sv_5 = v_6 \leftrightarrow \exists v_0 \exists v_1[v_5 = v_0 \wedge v_6 = v_1 \wedge Sv_0 = v_1]]$ 4.22

It remains only to put these together using 3.20.

(13) $$\begin{array}{l} \vdash S0 = v_4 \leftrightarrow \exists v_5 \exists v_6[0 = v_5 \wedge v_4 = v_6 \wedge \\ \quad \exists v_0 \exists v_1[v_5 = v_0 \wedge v_6 = v_1 \wedge Sv_0 = v_1]] \quad (10), (12) \end{array}$$

(14) $$\begin{array}{l} \vdash S0 = v_4 \leftrightarrow \exists v_5 \exists v_6[\exists v_0[v_0 = v_5 \wedge 0 = v_0] \wedge v_4 = v_6 \wedge \\ \quad \exists v_0 \exists v_1[v_5 = v_0 \wedge v_6 = v_1 \wedge Sv_0 = v_1]] \quad (11), (13) \end{array}$$

(15) $$\begin{array}{l} \vdash v_2 + v_3 = v_4 \leftrightarrow \exists v_5 \exists v_6 \exists v_7[v_2 = v_5 \wedge v_3 = v_6 \wedge v_4 = v_7 \wedge \\ \quad \exists v_0 \exists v_1 \exists v_2[v_5 = v_0 \wedge v_6 = v_1 \wedge v_7 = v_2 \wedge v_0 + v_1 = v_2]] \quad (8), (9) \end{array}$$

(16) $$\begin{array}{l} \vdash v_2 + v_3 = S0 \leftrightarrow \exists v_4[\exists v_5 \exists v_6 \exists v_7[v_2 = v_5 \wedge v_3 = v_6 \wedge v_4 = v_7 \wedge \\ \quad \exists v_0 \exists v_1 \exists v_2[v_5 = v_0 \wedge v_6 = v_1 \wedge v_7 = v_2 \wedge v_0 + v_1 = v_2]] \\ \quad \wedge \exists v_5 \exists v_6[\exists v_0[v_0 = v_5 \wedge 0 = v_0] \wedge v_4 = v_6 \wedge \\ \quad \exists v_0 \exists v_1[v_5 = v_0 \wedge v_6 = v_1 \wedge Sv_0 = v_1]]] \quad (7), (14), (15) \end{array}$$

(17) $$\begin{array}{l} \vdash v_0 + v_1 \cdot v_1 = 0 \leftrightarrow \exists v_2[ \\ \quad \exists v_3 \exists v_4 \exists v_5[v_0 = v_3 \wedge v_1 \cdot v_1 = v_4 \wedge v_2 = v_5 \wedge v_3 + v_4 = v_5] \wedge \\ \quad \exists v_0[v_0 = v_2 \wedge 0 = v_0] \quad (1), (2), (3) \end{array}$$

(18) $$\begin{array}{l} \vdash v_0 + v_1 \cdot v_1 = 0 \leftrightarrow \exists v_2[ \\ \quad \exists v_3 \exists v_4 \exists v_5[v_0 = v_3 \wedge \exists v_5 \exists v_6 \exists v_7[v_1 = v_5 \wedge v_1 = v_6 \wedge v_4 = v_7 \wedge v_5 \cdot v_6 = v_7] \\ \quad \wedge v_2 = v_5 \wedge \exists v_0 \exists v_1 \exists v_2[v_0 = v_3 \wedge v_1 = v_4 \wedge v_2 = v_5 \wedge v_0 + v_1 = v_2]] \wedge \\ \quad \exists v_0[v_0 = v_2 \wedge 0 = v_0] \quad (17), (4), (6)] \end{array}$$

(19) $$\begin{array}{l} \vdash v_0 + v_1 \cdot v_1 = 0 \leftrightarrow \exists v_2[ \\ \quad \exists v_3 \exists v_4 \exists v_5[v_0 = v_3 \wedge \exists v_5 \exists v_6 \exists v_7[v_1 = v_5 \wedge v_1 = v_6 \wedge v_4 = v_7 \wedge \\ \quad \exists v_0 \exists v_1 \exists v_2[v_0 = v_5 \wedge v_1 = v_6 \wedge v_2 = v_7 \wedge v_0 \cdot v_1 = v_2] \\ \quad \wedge v_2 = v_5 \wedge \exists v_0 \exists v_1 \exists v_2[v_0 = v_3 \wedge v_1 = v_4 \wedge v_2 = v_5 \wedge v_0 + v_1 = v_2]] \wedge \\ \quad \exists v_0[v_0 = v_2 \wedge 0 = v_0] \quad (18), (5)] \end{array}$$

Thus we get, finally, using (16) and (19)

$$\begin{array}{l} \vdash \forall v_0 \forall v_1[v_0 + v_1 \cdot v_1 = 0 \wedge v_2 + v_3 = S0] \leftrightarrow \forall v_0 \forall v_1[ \\ \quad \exists v_4[\exists v_5 \exists v_6 \exists v_7[v_2 = v_5 \wedge v_3 = v_6 \wedge v_4 = v_7 \wedge \\ \quad \exists v_0 \exists v_1 \exists v_2[v_5 = v_0 \wedge v_6 = v_1 \wedge v_7 = v_2 \wedge v_0 + v_1 = v_2]] \\ \quad \wedge \exists v_5 \exists v_6[\exists v_0[v_0 = v_5 \wedge 0 = v_0] \wedge v_4 = v_6 \wedge \\ \quad \exists v_0 \exists v_1[v_5 = v_0 \wedge v_6 = v_1 \wedge Sv_0 = v_1]]]] \wedge \\ \quad \exists v_2[\exists v_3 \exists v_4 \exists v_5[v_0 = v_3 \wedge \exists v_5 \exists v_6 \exists v_7[v_1 = v_5 \wedge v_1 = v_6 \wedge v_4 = v_7 \wedge \\ \quad \exists v_0 \exists v_1 \exists v_2[v_0 = v_5 \wedge v_1 = v_6 \wedge v_2 = v_7 \wedge v_0 \cdot v_1 = v_2] \\ \quad \wedge v_2 = v_5 \wedge \exists v_0 \exists v_1 \exists v_2[v_0 = v_3 \wedge v_1 = v_4 \wedge v_2 = v_5 \wedge v_0 + v_1 = v_2]] \wedge \\ \quad \exists v_0[v_0 = v_2 \wedge 0 = v_0]] \end{array}$$

$\boxed{16}$ Let $\Gamma$ consist of the following sentences:

$$\exists v_0 \ldots \exists v_{n-1}\left[\bigwedge_{i<j<n}(\neg(v_i = v_j))\right] \quad \text{for each integer } n > 1;$$

$$\forall v_0(v_0 E v_0);$$

$$\forall v_0 \forall v_1[v_0 E v_1 \rightarrow v_1 E v_0];$$

$$\forall v_0 \forall v_1 \forall v_2[v_0 E v_1 \wedge v_1 E v_2 \rightarrow v_0 E v_2];$$

$$\forall v_0 \exists v_1[v_0 E v_1 \wedge (\neg(v_0 = v_1)) \wedge \forall v_2(v_0 E v_2 \rightarrow v_2 = v_0 \vee v_2 = v_1)].$$

$\boxed{17}$ See Lemma 4 in the solution for exercise 3.

$\boxed{18}$ $\exists v_1 \exists v_2[v_0 = v_1 \cdot v_1 + v_2 \cdot v_2]$.

$\boxed{19}$ We take $x$ to be $v_0$ and $y$ to be $v_1$. Let $\overline{M}$ be a structure and $a : \omega \rightarrow M$ be any assignment. First suppose that $\overline{M} \models \exists v_0 G v_0 v_1[a]$. Choose $u \in M$ so that $\overline{M} \models G v_0 v_1[a^0_u]$. So $(u, a_1) \in G^{\overline{M}}$. Hence $\overline{M} \models G v_1 v_0[a^{0\,1}_{a_1\,u}]$ and so $\overline{M} \models \exists v_1 G v_1 v_0[a^0_{a_1}]$; so $\overline{M} \models (v_0 = v_1 \wedge \exists v_1 G v_1 v_0)[a^0_{a_1}]$. So $\overline{M} \models \exists v_0(v_0 = v_1 \wedge \exists v_1 G v_1 v_0)[a]$.

Second, suppose that $\overline{M} \models \exists v_0(v_0 = v_1 \wedge \exists v_1 G v_1 v_0)[a]$. Choose $u \in M$ so that $\overline{M} \models (v_0 = v_1 \wedge \exists v_1 G v_1 v_0)[a^0_u]$ Thus $u = a_1$. Choose $w \in M$ so that $\overline{M} \models G v_1 v_0[a^{0\,1}_{a_1\,w}]$. Hence $(w, a_1) \in G^{\overline{M}}$, so $\overline{M} \models G v_0 v_1[a^0_w]$, hence $\overline{M} \models \exists v_0 G v_0 v_1[a]$.

$\boxed{20}$ Expand the language by adding an individual constant $c$. Let $\Gamma'$ consist of $\Gamma$ together with all sentences

$$\exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{i<j<m} \neg(v_i = v_j) \wedge \bigwedge_{i<m} cRv_i\right]$$

Clearly every finite subset of $\Gamma'$ has a model, so $\Gamma'$ itself has a model $(M, R, d)$. Clearly $(M, R)$ is as desired.