

## Review questions for the final

(corrected on November 24, 2014)

1. Determine whether the following formula is a sentential tautology:

$$([\{S_1 \rightarrow (\neg\neg S_2 \rightarrow S_2)\} \rightarrow \neg\neg S_0] \rightarrow S_3] \rightarrow S_4 \rightarrow [(S_0 \rightarrow S_3) \rightarrow S_4]$$

2. Prove that  $\neg$  is not enough by itself to define all other sentential connectives.

3. In the proof system for sentential logic prove that  $\vdash S_0 \leftrightarrow \neg\neg S_0$ .

4. Give a formula in disjunctive normal form equivalent to the following formula:

$$(S_0 \wedge S_1) \rightarrow (S_2 \leftrightarrow S_3)$$

5. In the language for  $(\omega, S, 0, +, \cdot)$ , prove that the following is not a term:

$$\langle +, v_0, +, v_0, v_1, v_2 \rangle.$$

6. In the language for  $(\omega, S, 0, +, \cdot)$ , prove that the following is not a formula:  $\langle \forall, v_0, =, v_1, v_0, v_2, v_3 \rangle$ .

7. In the language for  $(\omega, S, 0, +, \cdot)$ , describe a formula  $\varphi$  with at most  $v_0$  free such that for any assignment  $a : \omega \rightarrow \omega$ ,  $(\omega, S, 0, +, \cdot) \models \varphi[a]$  iff  $a_0$  is a prime.

8. In the language for  $(\mathbb{R}, +, \cdot, 0, 1, <)$  describe a sentence  $\varphi$  such that for any structure  $\overline{M}$  for this language,  $\overline{M} \models \varphi$  iff every positive element has a square root.

9. Indicate which occurrences of variables are free and which ones bound in the following formula:

$$\forall v_0(v_0 + v_1 = v_2 \vee \exists v_1(v_1 + v_0 = v_2 \wedge \forall v_2(v_3 = v_2))).$$

10. Find a formula in prenex normal form equivalent to the following formula:

$$\forall v_0[v_0 > 0 \rightarrow \exists v_2(v_1 = v_3 \rightarrow \exists v_4(v_0 = v_1))].$$

11. Prove that the following set of sentences is consistent:

$$\begin{aligned} & \{\forall v_0 \forall v_1 [Sv_0 = Sv_1 \rightarrow v_0 = v_1], \\ & \exists v_0 \exists v_1 (\neg(v_0 = v_1)), \\ & \forall v_0 \forall v_1 \forall v_2 (v_0 = v_1 \vee v_0 = v_2 \vee v_1 = v_2)\}. \end{aligned}$$

12. Without using the completeness theorem, prove that  $\vdash \forall v_0 \exists v_1 \forall v_2 \varphi \rightarrow \forall v_0 \forall v_2 \exists v_1 \varphi$ .

13. Give an example to show that the restriction on  $\mathbf{c}$  in Lemma 4.9 is necessary.

14. Let  $\mathcal{L}$  be a language with a unary relation symbol  $F$  and a binary relation symbol  $G$ . Using the completeness theorem, prove that

$$\begin{aligned} & \vdash \exists x Fx \wedge \forall x \forall y [Fx \wedge Fy \rightarrow x = y] \rightarrow \\ & \quad [\exists x (Fx \wedge Gxy) \leftrightarrow \forall x (Fx \rightarrow Gxy)] \end{aligned}$$

15. Give a formula in standard form equivalent to the following formula:

$$\forall v_0 \forall v_1 [v_0 + v_1 \cdot v_1 = 0 \wedge v_2 + v_3 = S0].$$

16. Describe a set  $\Gamma$  of sentences in a language with a binary relation symbol such that the models of  $\Gamma$  are all pairs  $(A, E)$  such that  $A$  is infinite,  $E$  is an equivalence relation on  $A$ , and every equivalence class has exactly two elements.

Additional problems for math 5000 students

17. Using the proof system for sentential logic, prove that  $\vdash S_0 \rightarrow (S_1 \rightarrow S_0 \wedge S_1)$ .
18. In the language for  $(\omega, S, 0, +, \cdot)$  give a formula  $\varphi$  such that for any assignment  $a : \omega \rightarrow \omega$ ,  $(\omega, S, 0, +, \cdot) \models \varphi[a]$  iff  $a_0$  is a sum of two squares.
19. In a language with a binary relation symbol  $G$ , use the completeness theorem to prove that  $\vdash \exists x Gxy \leftrightarrow \exists x [x = y \wedge \exists y Gyx]$ .
20. In a language with one binary relation symbol  $R$ , suppose that  $\Gamma$  is a set of sentences containing sentences saying that  $R$  is an equivalence relation, such that for every natural number  $m$ ,  $\Gamma$  has a model with some equivalence class which has at least  $m$  elements. Show that  $\Gamma$  has a model with an infinite equivalence class.