Some details on the proofs of 4.29 and 4.30

We begin after (2) on page 79. The following formula is a tautology:

$$\neg(v_1 \in v_0) \to [\neg(v_1 \in v_2) \to (v_1 \in v_0 \leftrightarrow v_1 \in v_2)].$$

In fact, this formula is obtained from the following sentential tautology:

$$\neg S_0 \to (\neg S_1 \to (S_0 \leftrightarrow S_1)).$$

That this is a sentential tautology is seen as in chapter 1:

2	3	1	3	4	2	5	3	5
$\neg$	S_0	$\to$	$(\neg$	S_1	$\to$	$(S_0$	$\leftrightarrow$	$S_1))$
1	0	0	1	0	0	0	0	0

The assignments to $\leftrightarrow$ give a contradiction.

Using generalization and (L2) we get

$$(*) \qquad \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to \forall v_1(v_1 \in v_0 \leftrightarrow v_1 \in v_2)].$$

By (2) and 3.28 we get

$$(**) \qquad \text{ZFC} \vdash \forall v_1(v_1 \in v_0 \leftrightarrow v_1 \in v_2) \to v_0 = v_2)$$

Now the following is a tautology:

$$(***) \qquad (*) \wedge (**) \to [\forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]].$$

In fact, this is a form of the sentential tautology

$$[S_0 \to (S_1 \to S_2)] \wedge (S_2 \to S_3) \to [S_0 \to (S_1 \to S_3)];$$

we check that this is a tautology:

5	4	8	6	9	2	10	4	11	1	3	2	7	3	7
$[S_0$	$\to$	$(S_1$	$\to$	$S_2)]$	$\wedge$	$(S_2$	$\to$	$S_3)$	$\to$	$[S_0$	$\to$	$(S_1$	$\to$	$S_3)]$
1	1	1	1	1	1	1	1	1	0	1	0	1	0	0

Different values have been assigned to S_3, so we have a tautology.

Now from $(*)$, $(**)$, and $(***)$ we get

$$\text{ZFC} \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2],$$

and hence by generalization and 3.38 we get

$$(3) \qquad \text{ZFC} \vdash \forall v_0(\neg(v_1 \in v_0)) \to \forall v_2[\forall v_i(\neg(v_1 \in v_2)) \to v_0 = v_2].$$

Now $(S_0 \to S_1) \to (S_0 \to S_0 \wedge S_1)$ is clearly a sentential tautology, so by (3) we get

$$\mathrm{ZFC} \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

This has the form $\mathrm{ZFC} \vdash \varphi \to \psi$. Hence $\mathrm{ZFC} \vdash \neg\psi \to \neg\varphi$, hence $\mathrm{ZFC} \vdash \forall v_0\neg\psi \to \forall v_0\neg\varphi$, hence $\mathrm{ZFC} \vdash \neg\forall v_0\neg\varphi \to \neg\forall v_0\neg\psi$, i.e., $\mathrm{ZFC} \vdash \exists v_0\varphi \to \exists v_0\psi$. Thus

$$\mathrm{ZFC} \vdash \exists v_0\forall v_1(\neg(v_1 \in v_0)) \to \exists v_0[\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

From (1) it then follows that

$$\mathrm{ZFC} \vdash \exists v_0[\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

which is the desired conclusion. □

Now we turn to the proof of 4.30. The arguments are clear through (1). Now we continue; here we make some changes in the original arguments.

By the extensionality axiom and successive applications of the change of bound variables theorem 3.25 we have

$$\begin{aligned}
&\mathrm{ZFC} \vdash \forall v_0\forall v_1[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_1) \to v_0 = v_1];\\
&\mathrm{ZFC} \vdash \forall v_0\forall v_4[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_4) \to v_0 = v_4];\\
&\mathrm{ZFC} \vdash \forall v_0\forall v_4[\forall v_3(v_3 \in v_0 \leftrightarrow v_3 \in v_4) \to v_0 = v_4];\\
&\mathrm{ZFC} \vdash \forall v_2\forall v_4[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4) \to v_2 = v_4].
\end{aligned}$$

Then by Corollary 3.28 twice we get

(2) $$\mathrm{ZFC} \vdash \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4) \to v_2 = v_4].$$

Now the following is a tautology:

$$(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to [(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to (v_3 \in v_2 \leftrightarrow v_3 \in v_4)].$$

In fact, this is an instance of

$$(S_0 \leftrightarrow S_1) \to [(S_2 \leftrightarrow S_1) \to (S_0 \leftrightarrow S_2),$$

which is clearly a tautology.

Hence by generalization and (L2) we get

$$\begin{aligned}
\vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to\\
&[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4)].
\end{aligned}$$

Using (2) it follows that

$$\begin{aligned}
\mathrm{ZFC} \vdash &\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to\\
&[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to v_2 = v_4].
\end{aligned}$$

Here we use the tautology above, after $(***)$.

By generalization and Proposition 3.38 we then obtain

$$\begin{aligned}\text{ZFC} \vdash & \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\ & \forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4].\end{aligned}$$

By the tautology at the top of page 2 we then have

$$\begin{aligned}\text{ZFC} \vdash & \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ & \forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4]].\end{aligned}$$

Generalization and (L2) yield

$$\begin{aligned}\text{ZFC} \vdash & \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \exists v_2[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ & \forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4]].\end{aligned}$$

(See the argument on page 2, lines 3 and 4.) Hence by (1) we have

$$\begin{aligned}\text{ZFC} \vdash & \exists v_2[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ & \forall v_4[\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4]].\end{aligned}$$

This is the desired result. □