

Solutions to assignment 7

E4.2 Let \mathcal{L} be a language with just one non-logical constant, a binary relation symbol \mathbf{R} . Let Γ consist of all sentences of the form $\exists v_1 \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ with φ a formula with only v_0 free. Show that Γ is inconsistent. Hint: take φ to be $\neg \mathbf{R}v_0v_0$.

By Theorem 3.27 we have

$$(1) \quad \Gamma \vdash \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0] \rightarrow [\mathbf{R}v_1v_1 \leftrightarrow \neg \mathbf{R}v_1v_1].$$

Now $[\mathbf{R}v_1v_1 \leftrightarrow \neg \mathbf{R}v_1v_1] \rightarrow \neg(v_0 = v_0)$ is a tautology, so from (1) we obtain

$$\Gamma \vdash \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0);$$

then generalization gives

$$\Gamma \vdash \forall v_1 [\forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0)].$$

Then by Proposition 3.39 we get

$$\Gamma \vdash \exists v_1 \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0).$$

But the hypothesis here is a member of Γ , so we get $\Gamma \vdash \neg(v_0 = v_0)$. Hence by Lemma 4.1, Γ is inconsistent.

Alternate proof (due to a couple of students). Suppose that Γ is consistent. By the completeness theorem let \bar{A} be a model of Γ . Taking φ to be $\neg \mathbf{R}v_0v_0$, we get $\bar{A} \models \exists v_1 \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0]$. Let $a : \omega \rightarrow A$ be any assignment. Then by Proposition 2.8(iv) there is a $b \in A$ such that $\bar{A} \models \forall v_0 [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0][a_b^1]$. By the definition of satisfaction of \forall , it follows that for any $c \in A$ we have $\bar{A} \models [\mathbf{R}v_0v_1 \leftrightarrow \neg \mathbf{R}v_0v_0][a_c^0 \ b^1]$. Hence $(c, b) \in \mathbf{R}^{\bar{A}}$ iff $(c, b) \notin \mathbf{R}^{\bar{A}}$, contradiction.

E4.3 Show that the first-order deduction theorem fails if the condition that φ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$.

$$\begin{aligned} \{v_0 = v_1\} &\vdash v_0 = v_1 \\ \{v_0 = v_1\} &\vdash \forall v_1 (v_0 = v_1) \\ \{v_0 = v_1\} &\vdash \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_2 \quad \text{by Theorem 3.27} \\ \{v_0 = v_1\} &\vdash v_0 = v_2. \end{aligned}$$

On the other hand, let \bar{A} be the structure with universe ω and define $a = \langle 0, 0, 1, 1, \dots \rangle$. Clearly $\bar{A} \not\models [v_0 = v_1 \rightarrow v_0 = v_2][a]$. Hence $\not\vdash v_0 = v_1 \rightarrow v_0 = v_2$ by Theorem 3.2.

E4.4 In the language for $\bar{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let τ be the term $v_0 + v_1 \cdot v_2$ and ν the term $v_0 + v_2$. Let a be the sequence $\langle 0, 1, 2, \dots \rangle$. Let ρ be obtained from τ by replacing the occurrence of v_1 by ν .

(a) Describe ρ as a sequence of integers.

- (b) What is $\rho^{\bar{A}}(a)$?
- (c) What is $\nu^{\bar{A}}(a)$?
- (d) Describe the sequence $a^1_{\nu^{\bar{A}}(a)}$ as a sequence of integers.
- (e) Verify that $\rho^{\bar{A}}(a) = \tau^{\bar{A}}(a^1_{\nu^{\bar{A}}(a)})$ (cf. Lemma 4.4.)

- (a) ρ is $v_0 + (v_0 + v_2) \cdot v_2$; as a sequence of integers it is $\langle 7, 5, 9, 7, 5, 15, 15 \rangle$.
- (b) $\rho^{\bar{A}}(a) = 0 + (0 + 2) \cdot 2 = 4$.
- (c) $\nu^{\bar{A}}(a) = 0 + 2 = 2$.
- (d) $a^1_{\nu^{\bar{A}}(a)} = \langle 0, 2, 2, 3, \dots \rangle$.
- (e) $\rho^{\bar{A}}(a) = 4$, as above; $\tau^{\bar{A}}(a^1_{\nu^{\bar{A}}(a)}) = 0 + 2 \cdot 2 = 4$.

E4.5 In the language for $\bar{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, let φ be the formula $\forall v_0(v_0 \cdot v_1 = v_1)$, let ν be the formula $v_1 + v_1$, and let $a = \langle 1, 0, 1, 0, \dots \rangle$.

- (a) Describe $\text{Subf}_{\nu}^{v_1} \varphi$ as a sequence of integers
- (b) What is $\nu^{\bar{A}}(a)$?
- (c) Describe $a^1_{\nu^{\bar{A}}(a)}$ as a sequence of integers.
- (d) Determine whether $\bar{A} \models \text{Subf}_{\nu}^{v_1} \varphi[a]$ or not.
- (e) Determine whether $\bar{A} \models \varphi[a^1_{\nu^{\bar{A}}(a)}]$ or not.

- (a) $\text{Subf}_{\nu}^{v_1} \varphi$ is $\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1)$; as a sequence of integers it is

$$\langle 4, 5, 3, 9, 5, 7, 10, 10, 7, 10, 10 \rangle.$$

- (b) $\nu^{\bar{A}}(a) = (v_1 + v_1)^{\bar{A}}(\langle 1, 0, 1, 0, \dots \rangle) = 0 + 0 = 0$.
- (c) $a^1_{\nu^{\bar{A}}(a)} = \langle 1, 0, 1, 0, \dots \rangle$.
- (d) $\bar{A} \models \text{Subf}_{\nu}^{v_1} \varphi[a]$ iff $\bar{A} \models [\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1)][\langle 1, 0, 1, 0, \dots \rangle]$ iff for all $a \in \omega$, $a \cdot (0 + 0) = 0 + 0$; this is true.
- (e) $\bar{A} \models \varphi[a^1_{\nu^{\bar{A}}(a)}]$ iff $\bar{A} \models [\forall v_0(v_0 \cdot v_1 = v_1)][\langle 1, 0, 1, 0, \dots \rangle]$ iff for all $a \in \omega$, $a \cdot 0 = 0$; this is true.

E4.7 Let \bar{A} be an \mathcal{L} -structure, with \mathcal{L} arbitrary. Define $\Gamma = \{\varphi : \varphi \text{ is a sentence and } \bar{A} \models \varphi[a] \text{ for some } a : \omega \rightarrow A\}$. Prove that Γ is complete and consistent.

Note by Lemma 4.4 that $\bar{A} \models \varphi[a]$ for some $a : \omega \rightarrow A$ iff $\bar{A} \models \varphi[a]$ for every $a : \omega \rightarrow A$. Let φ be any sentence. Take any $a : \omega \rightarrow A$. If $\bar{A} \models \varphi[a]$, then $\varphi \in \Gamma$ and hence $\Gamma \vdash \varphi$. Suppose that $\bar{A} \not\models \varphi[a]$. Then $\bar{A} \models \neg\varphi[a]$, hence $\neg\varphi \in \Gamma$, hence $\Gamma \vdash \neg\varphi$.

This shows that Γ is complete. Suppose that Γ is not consistent. Then $\Gamma \vdash \neg(v_0 = v_0)$ by Lemma 4.1. Then $\Gamma \models \neg(v_0 = v_0)$ by Theorem 3.2. Since \bar{A} is a model of Γ , it is also a model of $\neg(v_0 = v_0)$, contradiction.