# Solutions for assignment 5

**E3.2** *Prove that (L6) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\rho = \sigma)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\rho^{\overline{A}}(a) = \sigma^{\overline{A}}(a)$, so $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, hence $\overline{A} \models (\rho = \tau)[a]$.

**E3.3** *Prove that (L8) is universally valid, in the proof of Theorem 3.1.*

Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Assume that

$$\begin{aligned}
&\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})[a];\ \text{hence}\\
&\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}};\ \text{hence}\\
&\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}};\ \text{hence}\\
&\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a];
\end{aligned}$$

hence (L8) is universally valid.

**E3.5** *Indicate which occurrences of the variables are bound and which ones free for the following formulas.*

$\exists v_0(v_0 < v_1) \wedge \forall v_1(v_0 = v_1)$.
$v_4 + v_2 = v_0 \wedge \forall v_3(v_0 = v_1)$.
$\exists v_2(v_4 + v_2 = v_0)$.

First formula: the first and second occurrences of $v_0$ are bound, and the third one is free. The first occurrence of $v_1$ is free, and the other two are bound.

Second formula: the occurrence of $v_3$ is bound. All other occurrences of variables are free.

Third formula: the two occurrences of $v_2$ are bound. The other occurrences of variables are free.

**E3.6** *Finish the proof of Proposition 3.13.*

Suppose that $\varphi$ is an atomic non-equality formula; so there is a relation symbol $\mathbf{R}$ and terms $\sigma_0, \ldots \sigma_{n-1}$ such that $\varphi$ is $\langle \mathbf{R}\rangle^\frown\sigma_0^\frown \cdots ^\frown \sigma_{n-1}$. Hence $i > 0$, and it is inside some term $\sigma_j$. By Proposition 3.12 there is a term which is a segment of $\sigma_j$ beginning at $i$; it is also a segment of $\varphi$, and it is unique by Proposition 2.2(iii).

Suppose inductively that $\varphi$ is $\neg\psi$, i.e., it is $\langle 0\rangle^\frown\psi$. Then $i > 0$, so that it is inside $\psi$. Hence the inductive hypothesis gives the desired result.

Suppose inductively that $\varphi$ is $\psi \to \chi$, i.e., it is $\langle 1\rangle^\frown\psi^\frown\chi$. Then $i > 0$ and $i$ is inside $\psi$ or $\chi$; the inductive hypothesis gives the desired result.

Suppose inductively that $\varphi$ is $\forall v_k\psi$, i.e., it is $\langle 4, 5(k+1)\rangle^\frown\psi$. So $i > 0$. If $i = 1$, then $\varphi_i$ is $5(k+1)$, so that $\langle 5(k+1)\rangle$ is a term which is a segment of $\varphi$, unique by Proposition 2.2(iii). If $i > 1$, then it is inside $\psi$, and the inductive hypothesis gives the desired result.

**E3.7** *Indicate all free and bound occurrences of terms in the formula $v_0 = v_1 + v_1 \to \exists v_2(v_0 + v_2 = v_1)$.*

$v_0$ is free in both of its occcurrences.
$v_1$ is free in all three of its occurrences.
$v_2$ is bound in both of its occurrences.
$v_1 + v_1$ is free in its occurrence.
$v_0 + v_2$ is bound in its occurrence.