Solutions for exercises in chapter 1

E1.1 *Verify that*

$$S_0 \to \neg S_1 = \langle 2,3,1,4\rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2,2,3,4,2,1,4,1,3\rangle.$$

$$\begin{aligned} S_0 \to \neg S_1 &= \langle 2\rangle^\frown S_0^\frown \neg S_1 \\ &= \langle 2\rangle^\frown \langle 3\rangle^\frown \langle 1\rangle^\frown S_1 \\ &= \langle 2,3,1,4\rangle; \end{aligned}$$

$$\begin{aligned} (S_0 \to S_1) \to (\neg S_1 \to \neg S_0) &= \langle 2\rangle^\frown (S_0 \to S_1)^\frown (\neg S_1 \to \neg S_0) \\ &= \langle 2\rangle^\frown \langle 2\rangle^\frown S_0^\frown S_1^\frown \langle 2\rangle^\frown \neg S_1^\frown \neg S_0 \\ &= \langle 2,2,3,4,2\rangle^\frown \langle 1\rangle^\frown S_1^\frown \langle 1\rangle^\frown S_0 \\ &= \langle 2,2,3,4,2,1,4,1,3\rangle. \end{aligned}$$

E1.3 *Prove that there is a sentential formula of each positive integer length.*

If m is a positive integer, then

$$\langle \overbrace{1,1,\ldots,1}^{m-1 \text{ times}}, S_0\rangle$$

is a formula of length m, it is

$$\overbrace{\neg\neg\cdots\neg}^{m-1 \text{ times}} S_0.$$

E1.5 *Prove Proposition 1.3 as follows. Let f be a sentential assignment. For each positive integer m, let A_m be the set of all sentential formulas of length at most m. An m-approximation is a function G assigning to each member of A_m a value 0 or 1 so that the following conditions hold:*

(1) If $S_i \in A_m$, then $G(S_i) = f(i)$.
(2) If $\neg\varphi \in A_m$, then $G(\neg\varphi) = 1 - G(\varphi)$.
(3) If $\varphi \to \psi$ is in A_m, then $G(\varphi \to \psi) = 0$ iff $G(\varphi) = 1$ and $G(\psi) = 0$.

Prove:

(4) If G and G' are m-approximations, then $G = G'$.

(5) For each positive integer m there is an m-approximation.

Then one can define the desired function F by setting $F(\varphi) = G(\varphi)$ where G is an m-approximation with φ of length m.

Following the outline, to prove (4), suppose that G and G' are m-approximations. We prove by induction on $i \le m$ that if φ is a formula of length i, then $G(\varphi) = G'(\varphi)$.

Suppose that we know the result for formulas φ of length less than i, and ψ has length i, where $1 \leq i \leq m$. By Proposition 1.2(ii) we have three cases.

Case 1. ψ is S_j for some j. Then $G(\psi) = G(S_j) = f(j) = G'(S_j) = G'(\psi)$.

Case 2. ψ is $\langle 0\rangle^\frown\chi$ for some formula χ. Thus the length of χ is $i-1 < i$, so $G(\chi) = G'(\chi)$ by the inductive assumption. Hence $G(\psi) = 1 - G(\chi) = 1 - G'(\chi) = G'(\psi)$.

Case 3. ψ is $\langle 1\rangle^\frown\chi^\frown\theta$ for some formulas χ, θ. Then the lengths of χ and θ are less than i, and so $G(\chi) = G'(\chi)$ and $G(\theta) = G'(\theta)$ by the inductive assumption. Hence

$$\begin{aligned} G(\psi) = 0 \quad &\text{iff} \quad G(\chi \to \theta) = 0 \\ &\text{iff} \quad G(\chi) = 1 \text{ and } G(\theta) = 0 \\ &\text{iff} \quad G'(\chi) = 1 \text{ and } G'(\theta) = 0 \\ &\text{iff} \quad G'(\chi \to \theta) = 0 \\ &\text{iff} \quad G'(\psi) = 0. \end{aligned}$$

and it follows that $G(\psi) = G'(\psi)$.

This finishes the inductive proof.

We prove (5) by induction. For $m = 1$, define $G(S_i) = f(i)$ for all $i \in \omega$. Clearly G is a 1-approximation. Now assume that we know that there is an n-approximation for every $n < m$, where $m > 1$. For each $n < m$ let G_n be an n-approximation. Let $\varphi \in A_m$. If φ has length $n < m$, let $H(\varphi) = G_n(\varphi)$. Now suppose that φ has length m. By Proposition 1.2(ii) we have the following cases:

Case 1. $\varphi = S_i$ for some i. But S_i has length 1 and $m > 1$, contradiction.

Case 2. $\varphi = \neg\psi$ for some formula ψ. Then ψ has length $m - 1$. We define $H(\varphi) = 1 - G_{m-1}(\psi)$.

Case 3. $\varphi = \psi \to \chi$ for some formulas ψ, χ. Say ψ has length $n < m$ and χ has length $p < m$. We define $H(\varphi) = 0$ iff $G_n(\psi) = 1$ and $G_p(\chi) = 0$.

Clearly H is an m-approximation.

E1.7 *Use the truth table method to show that the formula*

$$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$$

is a tautology.

φ	ψ	$\varphi \to \psi$	$\neg\varphi$	$\neg\varphi \vee \psi$	$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

E1.10 *Use the informal method described in the notes to determine whether or not the following is a tautology:*

$$S_0 \to (S_1 \to (S_2 \to (S_3 \to S_1))).$$

2	1	3	2	4	3	5	4	5
S_0	$\rightarrow$	$(S_1$	$\rightarrow$	$(S_2$	$\rightarrow$	$(S_3$	$\rightarrow$	$S_0)))$
1	0	1	0	1	0	1	0	0

Values 0 and 1 have been tentatively assigned to S_0, a contradiction, so the formula is a tautology.