11.5 Tensor algebras, Symmetric and Exterior Algebras.

Q: Can we make an R-module M into a ring.

Let R be a commutative ring with 1 and M an R-module.
We extend M to obtain a ring with addition $+$ and multiplication $\otimes$.

Definition. For $k \in \mathbb{N}$, define the set of *k-tensors*,

$$T^k(M) := M \otimes_R M \otimes_R \cdots \otimes_R M \quad (k \text{ factors}),$$
$$T^0(M) := R,$$
$$T(M) := \bigoplus_{k \geq 0} T^k(M).$$

M is a submodule of $T(M)$ by identifying M with $T^1(M)$.

Theorem.

(1) $(T(M), +, \otimes)$ *is an R-algebra (called the* tensor algebra of M*).* free R-algebra generated by M

(2) (Universal Property) *For any R-algebra A and R-module homomorphism $\varphi \colon M \to A$, there exists a unique R-algebra homomorphism $\Phi \colon T(M) \to A$ such that $\Phi|_M = \varphi$.*

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "\varphi"'] & T(M) \arrow[d, dotted, "\Phi"] \\ & A \end{tikzcd}$$

Proof. (2) R-multilinear $M^k \to A$, $(m_1, \ldots, m_k) \mapsto \varphi(m_1) \cdots \varphi(m_k)$, yields Φ. $\square$

Corollary. *Let V be a vector space over F with basis $B = \{v_1, \ldots, v_n\}$. Then*

$$v_{i_1} \otimes \cdots \otimes v_{i_k} \quad \text{for } i_1, \ldots, i_k \in \{1, \ldots, n\}$$

form a vector space basis for $T^k(V)$. In particular $\dim T^k(V) =$ n^k.

Proof. $\square$

Note. $T(V)$ can be regarded as the algebra of polynomials over F in noncommuting variables $v_1, \ldots, v_n$.

Example. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z}$.

$M \otimes M \cong$ $\mathbb{Z}/n\mathbb{Z} \cong M$

$T^k(M) \cong$ M for $k > 0$

$T(M) \cong$ $\mathbb{Z} \oplus M \oplus M \oplus \cdots$ as module

$\cong \mathbb{Z}[x]/(nx)$ as ring   identifying $1 \otimes 1$ with x

Symmetric Algebras.

Question. What is the maximal commutative quotient of $T(M)$?

Definition. The *symmetric algebra* of an R-module M is

$$S(M) := T(M)/C(M)$$

for the ideal $C(M)$ of $T(M)$ generated by $m_1 \otimes m_2 - m_2 \otimes m_1$ for all $m_1, m_2 \in M$.

Theorem.

(1) $S(M)$ *is a commutative R-algebra.*

(2) (Universal Property for maps to commutative R-algebras) *For any commutative R-algebra A and R-module homomorphism $\varphi \colon M \to A$, there exists a unique R-algebra homomorphism $\Phi \colon S(M) \to A$ such that $\Phi|_M = \varphi$.*

(3) (Universal Property for symmetric multilinear maps) *For any symmetric k-multilinear map $\varphi \colon M^k \to N$, there exists a unique R-module homomorphism $\Phi \colon S^k(M) \to N$ such that $\varphi = \Phi \circ \iota$ where*

$$\iota \colon M^k \to S^k(M), \ (m_1, \ldots, m_k) \mapsto m_1 \otimes \cdots \otimes m_k + C(M).$$

Proof. As in the general case using that $\underbrace{C^k(M)}_{C(M) \cap T^k(M)}$ is in the kernel of any symmetric map $T^k(M) \to N$. ☐

Corollary. *Let V be an n-dimensional F-vector space. Then $S(V) \cong$* $F[x_1, \ldots, x_n]$

Exterior Algebras.

Question. How to characterize alternating multilinear maps?

Definition. The *exterior algebra* of an R-module M is

$$\bigwedge(M) := T(M)/A(M)$$

for the ideal $A(M)$ of $T(M)$ generated by $m \otimes m$ for all $m \in M$.

We write

$$m_1 \wedge \cdots \wedge m_k := m_1 \otimes \cdots \otimes m_k + A(M)$$

and $\wedge$ for the product (called the *wedge* or *exterior product*) in $\bigwedge(M)$.

Note. Since $m \wedge m = 0$ for all $m \in M$,

(1) $m_1 \wedge m_2 = -m_2 \wedge m_1$ for all $m_1, m_2 \in M$ and
(2) $m_1 \wedge \cdots \wedge m_k = 0$ if $m_i = m_j$ for some $i \neq j$.

ad 1) $0 = (m_1+m_2) \wedge (m_1+m_2) = \underbrace{m_1 \wedge m_1}_{=0} + m_1 \wedge m_2 + m_2 \wedge m_1 + \underbrace{m_2 \wedge m_2}_{=0}$

Q: Does (1) imply $m \wedge m = 0 \ \forall m \in M$?
$m \wedge m = -m \wedge m \Rightarrow 2\, m \wedge m = 0$ yes if char $R \neq 2$

Theorem. (Universal Property for alternating multilinear maps) *For any alternating k-multilinear map $\varphi\colon M^k \to N$, there exists a unique R-module homomorphism $\Phi\colon \bigwedge^k(M) \to N$ such that $\varphi = \Phi \circ \iota$ where*

$$\iota\colon M^k \to \bigwedge^k(M),\ (m_1, \ldots, m_k) \mapsto m_1 \wedge \cdots \wedge m_k.$$

Corollary. *Let V be a vector space over F with basis $B = \{v_1, \ldots, v_n\}$. Then*

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \quad \textit{for } 1 \leq i_1 < \cdots < i_k \leq n$$

form a vector space basis for $\bigwedge^k(V)$ if $k \leq n$, $\dim \Lambda^k(V) = \binom{n}{k}$

If $k > n$, then $\Lambda^k(V) = 0$.

Homomorphisms of Tensor Algebras.

$\varphi \in \mathrm{Hom}_R(M, N)$ induces an R-module homomorphism on the k-th tensor power

$$T^k(\varphi)\colon T^k(M) \to T^k(N),\ m_1 \otimes \cdots \otimes m_k \mapsto \varphi(m_1) \otimes \cdots \otimes \varphi(m_k).$$

- Similar for S^k and $\bigwedge^k$. Since $C^k(M)$, $A^k(M)$ are invariant under $T^k(\varphi)$.
- Further φ induces R-algebra homomorphisms on $T(M), S(M), \bigwedge(M)$.

Example. Let V be an F-vector space with basis $B = \{v_1, \ldots, v_n\}$ and $\varphi \in \mathrm{End}_F(V)$.

Then $v_1 \wedge \cdots \wedge v_n$ is a basis of $\bigwedge^n V$ and

$$\bigwedge{}^n(\varphi)(v_1 \wedge \cdots \wedge v_n) = D(\varphi) v_1 \wedge \cdots \wedge v_n$$

for some $D(\varphi) \in F$.

For $A = (a_{ij}) \in M_{n\times n}(F)$ define $\varphi_A\colon V \to V$ such that $M_B^B(\varphi_A) = A$, i.e.,

$$\varphi_A(v_j) := \sum_{i=1}^n a_{ij} v_i.$$

Show that $D(A) := D(\varphi_A)$ satisfies the defining properties of the determinant to obtain the following.

Theorem. *Let* $\varphi \in \mathrm{End}_F(V)$ *and* $\dim_F V = n$. *Then*

$$\bigwedge{}^n(\varphi)(x) = (\det \varphi)x \text{ for all } x \in \bigwedge{}^n(V).$$

Note. Hence $\det \varphi$ is characterized as a *natural linear* map on $\bigwedge^n(V)$ *independent* of the choice of a basis B for $M_B^B(\varphi)$.