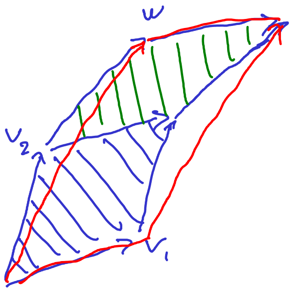
11.4 Determinants.

Let $A \in M_{n\times n}(\mathbb{R})$ have columns $v_1, \dots, v_n$.

Question. What is the 'volume' of the n-dimensional parallelepiped spanned by $v_1, \dots, v_n$,

$$\{\sum_{i=1}^{n} c_i v_i \ : \ c_i \in [0,1]\}?$$



Idea. A volume $D\colon R^n \times \cdots \times R^n \to R$ should have the following properties (R a commutative ring with 1):

(D1) *n-multilinear*: $\forall i \le n \ \forall v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n \in R^n$

$$R^n \to R, \ x \mapsto D(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n),$$

is an R-module homomorphism.

(D2) *alternating*:

$$D(v_1, \dots, v_n) = 0 \text{ whenever } v_i = v_j \text{ for } 1 \le i < j \le n$$

(D3) normalized: $D(e_1, \dots, e_n) = 1$.

Lemma. *Let $D\colon R^n \times \cdots \times R^n \to R$ be multilinear and alternating, let $v_1, \dots, v_n \in R^n$ and $1 \le i \ne j \le n$. Then*

(1) $D(\dots, v_j, \dots, v_i, \dots) = -D(\dots, v_i, \dots, v_j, \dots)$,

(2) $D(v_{\pi(1)}, \dots, v_{\pi(n)}) = \epsilon(\pi) D(v_1, \dots, v_n)$ *for all* $\pi \in S_n$ *with sign* $\epsilon(\pi)$,

(3) $D(v_1, \dots, v_{i-1}, v_i + c v_j, v_{i+1}, \dots, v_n) = D(v_1, \dots, v_n)$ *for all* $c \in R$.

Proof.

1) Let $\varphi(x,y) := D(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_{j-1}, y, v_{j+1}, \dots, v_n)$

Show $\varphi(y,x) = -\varphi(x,y)$.

$$0 = \varphi(x+y, x+y) = \varphi(x, x+y) + \varphi(y, x+y) = \underbrace{\varphi(x,x)}_{=0} + \varphi(x,y) + \varphi(y,x) + \underbrace{\varphi(y,y)}_{=0}$$

2) Let $\pi \in S_n$, say a product of k transpositions, i.e., $\epsilon(\pi) = (-1)^k$.

Apply 1) k times

3) immediate. □

Lemma. *Let $D\colon R^n \times \cdots \times R^n \to R$ be multilinear and alternating, let $v_1, \dots, v_n, w_1, \dots, w_n \in R^n$ with $a_{ij} \in R$ such that*

$$v_j = \sum_{i=1}^{n} a_{ij} w_i \text{ for all } j \le n.$$

Idea: $(a_{ij}) = M^B_B(\varphi)$ where $B = (w_1, \dots, w_n)$ basis of R^n, $\varphi : w_i \to v_i$

Then

$$D(v_1, \dots, v_n) = \sum_{\pi \in S_n} \epsilon(\pi)\, a_{\pi(1)1} \dots a_{\pi(n)n}\, D(w_1, \dots, w_n).$$

Proof. Expand $D(v_1, \dots, v_n)$ using multilinearity to get n^n summands $a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} D(w_{i_1}, \dots, w_{i_n})$ for $i_1, \dots, i_n \in \{1, \dots, n\}$.

These are 0 except if $i_1, \dots, i_n$ are all distinct, i.e., a permutation π of $\{1, \dots, n\}$. So

$$D(v_1, \dots, v_n) = \sum_{\pi \in S_n} a_{\pi(1)1} \cdots a_{\pi(n)n} D(w_{\pi(1)}, \dots, w_{\pi(n)})$$ □

Theorem. *The* determinant

$$\det\colon M_{n\times n}(R) \to R,\ (a_{ij}) \mapsto \sum_{\pi \in S_n} \epsilon(\pi)\, a_{\pi(1)1} \dots a_{\pi(n)n},$$

is the unique function $M_{n\times n}(R) \to R$ that is multilinear and alternating on the columns and satisfies $\det(I) = 1$.

Proof. Show det satisfies D1, D2, D3.

D1)

$$A = \begin{pmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_{i-1} & \alpha x + \beta y & v_{i+1} & \cdots & v_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

$B = (v_1 \cdots v_{i-1}\ x\ v_{i+1} \cdots v_n)$

$C = (\qquad y \qquad)$

$$\det A = \sum_{\pi \in S_n} \epsilon(\pi)\, a_{\pi(1)1} \cdots \underbrace{a_{\pi(i)i}}_{} \cdots a_{\pi(n)n}$$

$$a_{\pi(i)i} = \alpha\, b_{\pi(i)i} + \beta\, c_{\pi(i)i}$$

$$= \alpha \det B + \beta \det C$$

etc.

Previous Lemma yields for $A = \begin{pmatrix} \vdots & & \vdots \\ v_1 & \cdots & v_n \\ \vdots & & \vdots \end{pmatrix}$, $v_j = \sum_{i=1}^{n} a_{ij} e_i$ for unit vectors e_i that

$$D A = \sum \epsilon(\pi)\, a_{\pi(1)1} \cdots a_{\pi(n)n} \underbrace{D(e_1, \dots, e_n)}_{=1} = \det(A)$$

So $D = \det$ is the unique function satisfying D1, D2, D3. □

A_{ij} is a minor of A

Theorem (Cofactor expansion along row i). *For $A = (a_{ij}) \in M_{n\times n}(R)$, let $A_{ij} \in M_{n-1\times n-1}(R)$ be obtained from A by deleting row i and column j. Then*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

for all $i \leq n$.

Proof. Show right side satisfies D1, D2, D3 by induction on n. □

Corollary (Adjugate). *The* adjugate *of $A \in M_{n\times n}(R)$ is defined as $n \times n$-matrix*

$$\operatorname{adj}(A) := ((-1)^{i+j} \det A_{ij})^T$$

and satisfies

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = (\det A) I_n.$$

I_n .. $n \times n$ identity matrix

Proof book, p 440.

Corollary. *For $A, B \in M_{n\times n}(R)$*

(1) $\det A^T = \det A$,
(2) $\det AB = \det A \cdot \det B$,
(3) $A \in \mathrm{GL}_n(R)$ *iff* $\det A$ *is a unit in* R.

Proof. 1) Let $A = (a_{ij})$

$$\det A^T = \sum_{\pi \in S_n} \epsilon(\pi)\, a_{1\pi(1)} \cdots a_{n\pi(n)}$$

$$= \sum_{\pi \in S_n} \epsilon(\pi)\, a_{\pi^{-1}(1)1} \cdots a_{\pi^{-1}(n)n}$$

$$= \sum_{\pi^{-1} \in S_n} \epsilon(\pi^{-1})\, a_{\pi^{-1}(1)1} \cdots a_{\pi^{-1}(n)n} = \det A$$

2) Let $C = A \cdot B$

Column j of C: $C_j = \sum_{i=1}^{n} b_{ij} A_i$

By the Lemma above $\det C = \det B \cdot \det A$

3) $\Rightarrow$ follows from 2) using $\det I_n = 1$

$\Leftarrow$ by adjugate. □

Note.

(1) Determinants of large $n \times n$-matrices are best computed by row reduction, matrix decomposition,...
Known algorithms run in time polynomial in n. $O(n^3)$

(2) Similar to the determinant, the *permanent* of a matrix $A \in M_{n\times n}(R)$ is multilinear and symmetric in the columns,

$$\operatorname{perm} A := \sum_{\pi \in S_n} a_{\pi(1)1} \dots a_{\pi(n)n}.$$

perm A has applications in combinatorics, graph theory, ...

(3) Computation of the permanent is $\sharp$P-complete.
In particular, if there is a polynomial time algorithm for perm, then P=NP.