11.3 Dual spaces.

Example. $V := \{f : \mathbb{R} \to \mathbb{R} : \ f \text{ differentiable } \}$ is a vector space over $\mathbb{R}$

$$\left.\begin{array}{l} f \mapsto f(1) \\ f \mapsto f'(0) \\ f \mapsto \int_0^1 f(x)\,dx \end{array}\right\} \in \mathrm{Hom}(V, \mathbb{R})$$

Definition. $V^* := \mathrm{Hom}_F(V, F)$ is the *dual space* of the F-vector space V; its elements are *linear functionals.* (linear forms)

If $B = \{b_1, \dots, b_n\}$ a basis of V, then $B^* := \{b_1^*, \dots, b_n^*\}$ defined by

$$b_i^*(b_j) := \delta_{ij} \text{ for } i, j \in \{1, \dots, n\}$$

Kronecker delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$

is the *dual basis* of B. $b_i^* : V \to F$

$$\mathrm{Hom}_F(V, F) \cong M_{1\times n}(F)$$

Theorem. *Let $B = \{b_1, \dots, b_n\}$ be a basis of V.*

(1) $\dim V^* = n$
(2) $M_B^{E_1}(b_i^*) = (0 \dots 0 \overset{i}{1} 0 \dots 0)$
(3) *B^* is a basis of V^*.*

Question. What is the dual of the standard basis E_n of F^n?

$$e_i^*\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = x_i \qquad \text{projection on } i\text{-th component.}$$

$$V \xrightarrow{\varphi} W \xrightarrow{f} F$$
$$V^* \xleftarrow{\varphi^*} W^*$$

Fix $\varphi \in \mathrm{Hom}_F(V, W)$.
For $f \in \mathrm{Hom}_F(W, F)$, $f \circ \varphi \in \mathrm{Hom}_F(V, F)$. Define

$$\varphi^*\colon W^* \to V^*, \; f \mapsto f \circ \varphi.$$

Theorem. *Let V, W be finite dimensional with bases B, C, respectively. Let $\varphi \in \mathrm{Hom}_F(V, W)$.*
Then $\varphi^* \in \mathrm{Hom}_F(W^*, V^*)$ and $M_{C^*}^{B^*}(\varphi^*) = M_B^C(\varphi)^T$.

Proof. φ^* is linear because $f, g \in W^*$, $c \in F$
$\varphi^*(f+g) = (f+g)\circ\varphi = f\circ\varphi + g\circ\varphi = \varphi^*(f) + \varphi^*(g)$
$\varphi^*(cf) =$

Let $B = (b_1, \dots, b_n)$, $C = (c_1, \dots, c_m)$ and $M_B^C(\varphi) = (a_{ij})$. Then
$\varphi(b_j) = \sum_{i=1}^m a_{ij} c_i$ for all $j \le n$.

Claim: $\varphi^*(c_i^*) = \sum_{j=1}^n a_{ij} b_j^*$. (Then $M_{C^*}^{B^*}(\varphi^*) = (a_{ji})$)

Evaluate both sides on B:
$(\varphi^*(c_i^*))(b_j) = (c_i^* \circ \varphi)(b_j) = c_i^*(\varphi(b_j)) = c_i^*\left(\sum_{k=1}^m a_{kj} c_k\right) =$
$= \sum_{j=1}^m a_{kj} \underbrace{c_i^*(c_k)}_{\delta_{ik}} = a_{ij}$
$\left(\sum_{k=1}^n a_{ik} b_k^*\right)(b_j) = \sum_{k=1}^n a_{ik} \underbrace{b_k^*(b_j)}_{\delta_{kj}} = a_{ij}$. Claim proved. □

Corollary. *For any matrix A, the column rank of A is equal to the row rank of A.*

Proof. Show $\varphi\colon x \mapsto Ax$ and φ^* have the same rank (dimension of image).
See book (page 436).

Note. If $\dim V < \infty$, then $V \cong V^* \cong (V^*)^*$.

While the isomorphism $V \to V^*$ depends on a choice of the basis,

$$V \to (V^*)^*, \; v \mapsto E_v,$$

natural embedding

for the *evaluation map* $E_v\colon V^* \to F$, $f \mapsto f(v)$, does not.

Note: If V has an infinite basis B, then $V = FB$ is the direct sum but $V^* \cong F^B$ direct product (HW).