11.2 Matrix of a linear transformation.

Let V, W be F-vector spaces with (ordered) bases $B = (b_1, \dots, b_n), C = (c_1, \dots, c_m)$, respectively. Let $\varphi \in \mathrm{Hom}_F(V, W)$. For $j \leq n$, let

$$\varphi(b_j) = \sum_{i=1}^{m} a_{ij} c_i$$

for coordinates $a_{ij} \in F$.

$$\overset{\text{basis of codomain}}{} \quad M_B^C(\varphi) := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \underset{\text{basis of domain}}{}$$

$m \times n$ - matrix with j-th column the coordinate tuple of $\varphi(b_j)$ w.r.t. C

is the *matrix of φ with respect to B, C.*

Example.

$\varphi : F^2 \to F^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 \\ x_2 \\ 2x_1 + x_2 \end{pmatrix}$

Let E_2, E_3 be standard basis for F^2, F^3, resp.

$[\varphi(e_1)]_{E_3} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\qquad M_{E_2}^{E_3}(\varphi) = \begin{pmatrix} 1 & \vdots \\ 0 & \vdots \\ 2 & \vdots \end{pmatrix}$

Note. If $v = \sum a_j b_j$, for $a_j \in F$, then $M_B^C(\varphi) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ are the coordinates of $\varphi(v)$ with respect to C.

Theorem. *Let V, W be F-vector spaces with bases B, C of size n, m, respectively. Then*

$$\mathrm{Hom}_F(V, W) \to M_{m \times n}(F), \; \varphi \mapsto M_B^C(\varphi),$$

is an F-vector space isomorphism.

Proof. Lin Algebra. □

Theorem. *Let U, V, W be vector spaces with bases A, B, C, respectively, let $\psi \in \mathrm{Hom}(U, V), \varphi \in \mathrm{Hom}(V, W)$. Then*

$$M_A^C(\varphi \circ \psi) = M_B^C(\varphi) \cdot M_A^B(\psi).$$

Corollary. *Let V be an F-vector space with basis B, $|B| = n$. Then*

(1) $\mathrm{End}_F(V, \cancel{V}) \to M_n(F), \; \varphi \mapsto M_B^B(\varphi)$, *is a ring ($F$-algebra) isomorphism.*

(2) $\mathrm{GL}(V) \cong \mathrm{GL}_n(F)$ *as groups.*

Example.

Similarity.
Let B, C be bases of V, $\dim V = n$, $\varphi \in \mathrm{End}_F(V, V)$.

Question. What is the connection between $M_B^B(\varphi)$ and $M_C^C(\varphi)$?

$M_C^C(\varphi) = M_B^C(id) \cdot M_B^B(\varphi) \cdot M_C^B(id)$

Note: $M_B^C(id) = M_C^B(id)^{-1}$

Definition. $A, B \in M_n(F)$ are *similar* if there exists $P \in \mathrm{GL}_n(F)$ such that

$$P^{-1}AP = B$$

Recall. $A, B \in M_n(F)$ are similar iff the $F[x]$-modules F^n under $x \cdot v = Av$ and $x \cdot v = Bv$ are isomorphic.

$V_A \cong V_B$ iff A, B similar

Question. How to classify similarity classes/$\mathrm{GL}_n(F)$-orbits of matrices?