11.1 Vector spaces.

Recall from Linear Algebra:

- A ring F is a *field* if $(F - \{0\}, \cdot)$ is a commutative group.
- An F-module V for a field F is a *vector space* over F (F-vector space).
- $A \subseteq V$ is *linearly independent* in an F-vector space V if $\forall n \in \mathbb{N}\ \forall$ distinct $a_1, \ldots, a_n \in A\ \forall r_1, \ldots, r_n \in F$:

$$r_1 a_1 + \ldots r_n a_n = 0 \text{ implies } r_1 = \cdots = r_n = 0.$$

- $A \subseteq V$ *spans* V if $FA = V$.
- A is a *basis* of $FA \leq V$ iff A is linearly independent.

Favorite **Example.**

1) F^n has basis $e_1, \ldots, e_n$

2) $F[x]$ has basis $\{x^i \mid i \in \mathbb{N}_0\}$

Question.

(1) Does every vector space V have a basis?
(2) If V has bases A, B, how do they relate?

Theorem. *Every finitely generated vector space has a basis.*

Proof. We have $V = FA$ for A finite.

If $\exists b \in A$ such that

$$\sum_{a \in A} c_a a = 0 \quad \text{with } c_b \neq 0.$$

Then $A \setminus \{b\} =: A_1$ still generates V

Repeat $A \supseteq A_1 \supseteq A_2 \ldots$

Process stops since A is finite at some basis A_n of V. $\square$

Example.

$\ast$ **Theorem.** *If A, B are bases of a vector space V and $|A|$ is finite, then $|A| = |B|$.*

follows from

Replacement Theorem. *Let $\{a_1, \ldots, a_n\}$ span V, let $\{b_1, \ldots, b_m\} \subseteq V$ linearly independent. Then $\exists \pi \in S_n\ \forall k \le m:\ \{b_1, \ldots, b_k, a_{\pi(k+1)}, \ldots, a_{\pi(n)}\}$ spans V. In particular, $m \le n$.*

Proof. by induction on k.

$k = 0$ ✓

Ind. hypothesis $\{b_1, \ldots, b_k, a_{\pi(k+1)}, \ldots, a_{\pi(n)}\}$ spans V

Then $b_{k+1} = \beta_1 b_1 + \ldots + \beta_k b_k + \alpha_{k+1} a_{\pi(k+1)} + \ldots + \alpha_n a_{\pi(n)}$

Since $b_1, \ldots, b_{k+1}$ is lin independent, there exist some $i \in \{k+1, \ldots, n\}$ s.t. $\alpha_i \neq 0$.

Redefining π if necessary, assume $\alpha_{k+1} \neq 0$.

Then $a_{\pi(k+1)} \in F\{b_1, \ldots, b_k, b_{k+1}, a_{\pi(k+2)}, \ldots, a_{\pi(n)}\} = V$ $\square$

Proof of Thm ($\ast$): By Replacement Thm, the size of any lin independent set is less or equal to the size of any spanning set. So $|A| \le |B|$ and $|B| \le |A|$. $\square$

Dimension.

Definition. If a vector space V is finitely generated, the *dimension* $\dim V$ is the size of a basis of V; V is *finite dimensional.*

If V is not finitely generated, then V is *infinite dimensional*, $\dim V = \infty$.

Example.

$\dim F^n$

$\dim F[x]$

Corollary (Building-Up Lemma)**.** *Let V be finite dimensional and $A \subseteq V$ linearly independent. Then there exists a basis of V containing A.*

Proof. Use Replacement Thm with basis B and A lin independent $\square$

Theorem. *If V is an F-vector space and $\dim V = n$, then $V \cong F^n$.*

Proof. Let $B = (b_1, \ldots, b_n)$ be a basis for V.

$\varphi: F^n \to V$ is an iso.

$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto \sum_{i=1}^n a_i b_i$ $\square$

Question. What is a basis of $\mathbb{R}^{\mathbb{N}}$?

unit vectors only span direct sum $\mathbb{R}^{(\mathbb{N})}$, not $\mathbb{R}^{\mathbb{N}}$

Note. That every infinite dimensional vector space has a basis (= a maximal linearly independent set) is equivalent to

Zorn's Lemma. If every chain in a partially ordered set P has an upper bound in P, then P contains a maximal element.

HW

Theorem. *Let V be a vector space over F with subspace W.*
Then $\dim V = \dim V/W + \dim W$ *(where if one side is infinite, then both are).*

Proof. $V/W = \{ v + W \mid v \in V \}$

see book

Corollary. *For* $\varphi \in \mathrm{Hom}_F(V, W)$,

$$\dim V = \dim \ker \varphi + \dim \varphi(V).$$

Proof. 1st Isomorphism Thm and previous Thm. □

* Application of Zorn's Lemma: Every ring with 1 has a max ideal.

Order the set P of proper ideals of R by $\subseteq$.

$I_1 \subseteq I_2 \subseteq \dots \quad \subseteq \bigcup I_i \neq R$ since $1 \notin I_i$ for any i.