General tensor product. For defining the generators of H and defining an S-module above we needed

- S is a right R-module and N is a left R-module,
- S is a left S-module with $s(s'r) = (ss')r$ for all $s, s' \in S, r \in R$.

We generalize these from S to M:

Definition. For rings R, S an *(S, R)-bimodule M* is a left S-module and a right R-module satisfying

$$s(mr) = (sm)r \quad \forall s \in S, r \in R, m \in M.$$

Example.

1) Any ring S is an (S,R)-bimodule for any $R \leq S$.

2) Let R be commutative, M a left R-module.
Then M is also a right R-module via
$mr := rm$ for $m \in M, r \in R$.
Further M is an (R,R)-bimodule.

Definition. Let M be an (S, R)-bimodule, N an R-module. The *tensor product* $M \otimes_R N$ of M and N over R is the quotient of the free $\mathbb{Z}$-module over $M \times N$ by the $\mathbb{Z}$-submodule H generated by

$$\begin{aligned}(m_1 + m_2, n) - (m_1, n) - (m_2, n)\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2)\\ (mr, n) - (m, rn)\end{aligned}$$

for $m, m_1, m_2 \in M, n, n_1, n_2 \in N, r \in R$.

Elements in $M \otimes_R N$ are called *tensors* and can be written (non-uniquely) as finite sums of 'simple' tensors $m \otimes n := (m, n) + H$ for $m \in M, n \in N$.

Lemma. *$M \otimes_R N$ is an S-module under*

$$s(\sum m_i \otimes n_i) := \sum (sm_i) \otimes n_i.$$

Proof as for Lemma above.

Important special case

$M \otimes_R N$ for $S = R$ commutative.
Recall: for R commutative any R-module M is an (R,R)-bimodule.

Hence for any R-module N, $M \otimes_R N$ is a left R-module with

$$r(m \otimes n) = rm \otimes n = mr \otimes n = m \otimes rn$$

for $r \in R, m \in M, n \in N$.

$\iota\colon M \times N \to M \otimes_R N$, $(m,n) \mapsto m \otimes n$, is additive in both components and satisfies

$$r\iota(m,n) = \iota(rm,n) = \iota(m,rn),$$

(i.e. ι is R-bilinear).

Definition. Let R be commutative and M, N, L be R-modules. Then $\varphi\colon M \times N \to L$ is R-*bilinear* if it is R-linear in both components.

Ex dot-product on VS is bilinear

Theorem. *Let R be commutative and M, N, L be R-modules. Then there is a bijection*

$$\left\{ \begin{array}{c} R-\textit{bilinear maps} \\ \varphi\colon M \times N \to L \end{array} \right\} \to \left\{ \begin{array}{c} R-\textit{module homomorphisms} \\ \Phi\colon M \otimes_R N \to L \end{array} \right\}$$

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

"$M \otimes_R N$ is universal with respect to bilinear functions $M \times N \to L$"

Proof.

1) For $\Phi \in \mathrm{Hom}_R(M \otimes N, L)$, $\Phi \circ \iota$ is bilinear.

2) For $\varphi : M \times N \to L$ bilinear over R

As in the proof of the previous Thm, φ vanishes on H and induces a unique $\mathbb{Z}$-mod hom

$\tilde{\varphi} : F(M \times N) \to L$, $(m,n) \mapsto \varphi(m,n)$

$H \subseteq \ker \tilde{\varphi}$ and $\tilde{\varphi}$ induces $\Phi : F(M \times N)/H \to L$,

which is an R-mod hom since

$$r\Phi(m \otimes n) = r\varphi(m,n) = \varphi(rm,n) = \Phi(rm \otimes n) = \Phi(r(m \otimes n))$$

∎

Example.

1) Let M, N be vectorspaces over a field F with bases B, C, resp.
Then $M \otimes_F N$ has basis $B \times C$.
$\dim M \otimes_F N = \dim M \cdot \dim N$

2) $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 \cong 0$.
$a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes 0 = 0$

3) $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is generated by $1 \otimes 0 = 0 \otimes 1 = 0 \otimes 0 = 0$ and
$1 \otimes 1 \neq 0$ since $\varphi: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $(x,y) \mapsto xy$ is bilinear, nontriv.
hence $\Phi(1 \otimes 1) \neq 0$.

4) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ (HW)
$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ $\quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$

In the following all modules are over a commutative ring R.

Theorem (Tensor product of homomorphisms). *Let $\varphi \in \mathrm{Hom}_R(M, M')$, $\psi \in \mathrm{Hom}_R(N, N')$. Then there exists a unique R-module homomorphism $\varphi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$ such that*

$$(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n) \quad \forall m \in M, n \in N.$$

Theorem.

(1) $M \otimes_R N \cong N \otimes_R M$
(2) $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$
(3) $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$
(4) $R^s \otimes_R R^t \cong R^{st}$
(5) *If $R \leq S$, then $S \otimes_R R^n \cong S^n$ as S-modules.*

Corollary. *There is a bijection between*

$$\left\{ \begin{array}{c} R - \text{multilinear maps} \\ \varphi : M_1 \times \cdots \times M_n \to L \end{array} \right\} \to \left\{ \begin{array}{c} R - \text{module homomorphisms} \\ \phi : M_1 \otimes_R \cdots \otimes_R M_n \to L \end{array} \right\}$$

Ex $\det : M_n(R) \to R$ is multilinear columns of matrices.