10.4 Tensor products.

Extending scalars. Let R be a subring of S (with the same 1).

- Each S-module is also an R-module.
- Conversely, can each R-module N be made an S-module?

No, the $\mathbb{Z}$-mod $\mathbb{Z}$ cannot be made into a $\mathbb{Q}$-mod.

Else $\underbrace{\frac{1}{2}\cdot 1}_{=:x} + \frac{1}{2}\cdot 1 = \left(\frac{1}{2}+\frac{1}{2}\right) 1 = 1\cdot 1 = 1$ yields $2x = 1$.

Still $\mathbb{Z}$ can be embedded into the $\mathbb{Q}$-module $\mathbb{Q}$.

We'd need a map $S \times N \to N,\ (s,n) \mapsto sn$, that is additive in both components and satisfies $(sr)n = s(rn)$ for all $s \in S, r \in R, n \in N$ to make the actions of R and S on N compatible.

Consider the free $\mathbb{Z}$-module (abelian group) over $S \times N$

$$F(S\times N) \qquad F := \{\sum_{i=1}^{k} a_i(s_i, n_i) \ :\ k \in \mathbb{N}, a_i \in \mathbb{Z}, s_i \in S, n_i \in N\}.$$

To obtain the properties of an S-module take the quotient of F by the subgroup H generated by all elements

$$\left.\begin{aligned} (s_1+s_2, n) - (s_1, n) - (s_2, n) \\ (s, n_1+n_2) - (s, n_1) - (s, n_2) \\ (sr, n) - (s, rn) \end{aligned}\right\} =: G$$

for $s, s_1, s_2 \in S, n, n_1, n_2 \in N, r \in R$.

Then $(s_1+s_2, n) \equiv (s_1, n) + (s_2, n)$ modulo H, etc.

Definition. $F/H =: S \otimes_R N$ is the *tensor product of S and N over R.*

Elements in $S \otimes_R N$ are called *tensors* and can be written (non-uniquely) as finite sums of 'simple' tensors $s \otimes n := (s,n) + H$ for $s \in S, n \in N$.

Lemma. *$S \otimes_R N$ is an S-module under*

$$s(\sum s_i \otimes n_i) := \sum (ss_i) \otimes n_i,$$

called the S-module obtained by extension of scalars from the R-module N.

Proof. Show the action of S is well defined.

1) Let $h = \sum (s_i, n_i) \in H$, $t \in S$.

Then $\sum (ts_i, n_i) \in \langle G \rangle = H$

Ex. $h = (s, n_1+n_2) - (s, n_1) - (s, n_2) + (sr, n) - (s, rn)$

$\underbrace{(ts, n_1+n_2) - (ts, n_1) - (ts, n_2)}_{\in G} + \underbrace{(tsr, n) - (ts, rn)}_{\in G}$

2) Let $\sum s_i \otimes n_i = \sum s_i' \otimes n_i'$ in $S \otimes_R N$

Then $\sum (s_i, n_i) - \sum (s_i', n_i') \in H$

By 1) $\sum (ts_i, n_i) - \sum (ts_i', n_i') \in H$

So $\sum ts_i \otimes n_i = \sum ts_i' \otimes n_i'$

Check the module properties

$(t+t')(s\otimes n) = t(s\otimes n) + t'(s\otimes n)$

etc

□

$\iota\colon N \to S\otimes_R N$, $n \mapsto 1\otimes n$, is an R-module homomorphism (not necessarily injective). nor surjective)

$S\otimes_R N$ is the 'best possible' S-module as codomain of an R-module homomorphism by the following *universal property*:

Theorem. *Let L be an S-module and $\varphi \in \mathrm{Hom}_R(N, L)$. Then there exists a unique $\Phi \in \mathrm{Hom}_S(S\otimes_R N, L)$ such that $\varphi = \Phi\circ\iota$.*

~~*Proof.*~~

$$\begin{tikzcd}
N \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & S\otimes_R N \arrow[d, dashed, "\Phi"] & F \arrow[dl, "\tilde\varphi"'] \\
& L & S\times N \arrow[l] \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

"φ factors through $S\otimes_R N$"

Prof. Let F is the free $\mathbb{Z}$-mod over $S\times N$.

By the universal mapping property of $\mathbb{Z}$-modules $\exists!$ $\mathbb{Z}$-mod hom

$\tilde\varphi: F \to L$, $(s,n) \mapsto s\varphi(n)$

Note $\tilde\varphi$ is 0 on G, hence $\tilde\varphi(H) = 0$.

So $\tilde\varphi$ induces a well defined $\Phi: F/H \to L$ with

$\Phi(s\otimes n) = \tilde\varphi(s,n) = s\varphi(n)$

Claim: $\Phi \in \mathrm{Hom}_S(S\otimes_R N, L)$

Φ is additive since $\tilde\varphi$ is

Let $t\in S$. Then $\Phi(ts\otimes n) = ts\,\varphi(n) = t\,\Phi(s\otimes n)$.

Claim: Φ is uniquely determined by its images on the generators $1\otimes n$, hence by $1\cdot\varphi(n) = \varphi(n)$, i.e. by φ. □

Corollary. *$N/\ker\iota$ is the unique largest quotient of N that embeds into an S-module.*

Example. 1) Let N be an R-mod

$$R \otimes_R N \cong N$$

Use $\varphi = \mathrm{id}$ in the above Thm, to see ι is an iso with inverse Φ.

2) A <u>fin</u> abelian group, $\mathbb{Z}$-mod

$$\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$$

Let $q \in \mathbb{Q}$, $n = |A|$, $a \in A$

$$q \otimes a = \left(\frac{q}{n} n\right) \otimes a = \frac{q}{n} \otimes \underbrace{na}_{=0} = \frac{q}{n}\,(\underbrace{1 \otimes 0}_{0}) = 0.$$

3) <u>Induction of modules over group rings</u>

Let R be comm. ring with 1, $H \leq G$ fin group.

N an RH-module.

Then $RG \otimes_{RH} N$ is the RG-mod induced by RH.