Kernels and images.

How to check that a homomorphism is injective or surjective?

Definition. Let $\varphi \in \mathrm{Hom}_R(M, N)$. Then

$$\ker \varphi := \{m \in M \ : \ \varphi(m) = 0\}$$

is the *kernel* of φ;

$$\varphi(M) := \{\varphi(m) \ : \ m \in M\}$$

is the *image* of φ.

Lemma. $\ker \varphi \leq M$, $\varphi(M) \leq N$

Recall: Kernels are typically special substructures

- in groups normal subgroups
- in rings ideals
- in vector spaces arbitrary subspaces

Quotients.

Lemma. *Let N be a submodule of an R-module M. Then the quotient group*

$$M/N := \{m + N \ : \ m \in M\}$$

is an R-module under

$$\underline{r(m+N) := rm + N} \text{ for } r \in R, m \in M.$$

Proof. $+$ is well defined since M is abelian.

Check: action of R on M/N is <u>well defined</u>

Let $x, y \in M$ s.t. $x + N = y + N$ i.e. $x - y \in N$

Let $r \in R$. Then $\underbrace{r(x-y)}_{rx - ry} \in N$

Hence $rx + N = ry + N$.

Module properties of M/N are inherited from M. $\square$

Note. For $N \leq M$ the projection (quotient map)

$$\pi \colon M \to M/N, \ x \mapsto x + N,$$

is an R-module homomorphism,

$\ker \pi = N$

$M/\ker \pi = M/N = \pi(M)$.

Hence any submodule is a kernel (like in vector spaces).

Example. For $i \leq n$ and the R-module R^n

$$\pi_i \colon R^n \to R, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto x_i,$$

is an R-module homomorphism;

$\ker \pi_i = R^{i-1} \times \{0\} \times R^{n-i}$

Considering R^n as $M_n(R)$-module, π_i is not an $M_n(R)$-mod hom since $\ker \pi_i$ is not closed under $M_n(R)$.

Definition. For $A, B \leq M$

$$A + B := \{a + b \;:\; a \in A, b \in B\}$$

$\cancel{\langle A \cup B \rangle}$

is the sum of A and B.

$A + B$ is the smallest R-submodule of M containing A and B.

Isomorphism Theorems. Let M, N be R-modules.

(1) Let $\varphi \in \mathrm{Hom}_R(M, N)$. Then $M/\ker\varphi \cong \varphi(M)$.

(2) For $A, N \leq M$

$$(N + A)/N \cong A/(N \cap A)$$

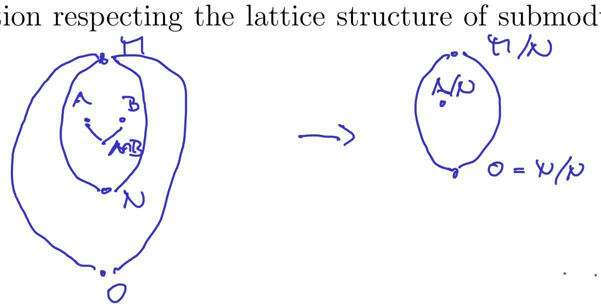
(3) For $N \leq A \leq M$

$$(M/N)\,/\,(A/N) \cong M/A$$

(4) For $N \leq M$

$$\{A \leq M \;:\; N \leq A\} \to \{R-\text{submodules of } M/N\},$$
$$A \mapsto A/N,$$

is a bijection respecting the lattice structure of submodules (intersection, sums).



Proof.

Use the results for abelian groups and check additional structure. $\square$

10.3 Generation of modules.

Definition. Let $1 \in R$, M an R-module. For $A \subseteq M$

$$RA := \{r_1 a_1 + \cdots + r_k a_k \;:\; k \in \mathbb{N}, r_1, \ldots, r_k \in R, a_1, \ldots, a_k \in A\}$$

finite R-linear combinations

is the smallest submodule containing A, called the *submodule of M generated by A*.

Note.

- If $A = \emptyset$, then $RA = \{0\}$ trivial submodule
- For $A = \{a_1, \ldots, a_n\}$ finite, $RA = Ra_1 + \cdots + Ra_n$.

Definition. $N \leq M$ is *finitely generated* if $N = RA$ for some finite set A; N is *cyclic* if $|A| = 1$.

Example.

1) The regular R-module R is cyclic, generated by 1.
2) The R-mod R^n is generated by $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$.
3) The $M_n(R)$-mod R^n is generated by $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Q: Does every nonzero vector in R^n generate R^n as $M_n(R)$-module?