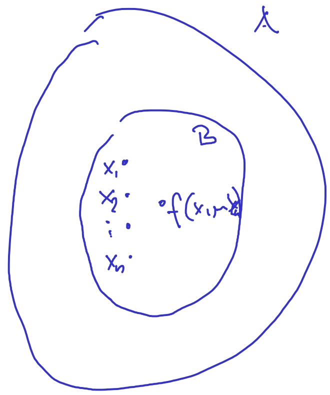
Definition. Let R be a commutative ring with 1. An *R-algebra* A is a ring with 1 which is also an R-module such that $\forall r \in R,\ \forall a, b \in A$

$$r(ab) = (ra)b = a(rb).$$

Example.

1) Rings are $\mathbb{Z}$-algebras
2) $M_n(R)$ is an R-module under
   $(rA)_{ij} := rA_{ij}$ componentwise multiplication

Substructures.
For any algebraic structure A, (e.g. group, ring, vectorspace, module)
a *substructure* B is a nonempty subset that is closed under all operations $f, ..$ of A.



Definition. N is an *R-submodule* of an R-module M (denoted $N \leq M$) if N is nonempty and closed under the operations of M, i.e.,

1) $(N,+)$ is a subgroup of $(M,+)$
2) $\forall r \in R\ \forall n \in N$: $rn \in N$

Example.

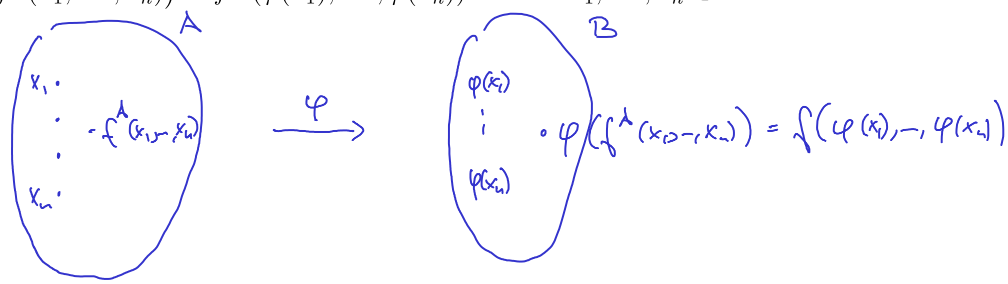
1) Let L be a left ideal of R.
Then L is an R-submodule of the regular R-module R
(and conversely).
2) R^n has R-submodules
$\left\{ \begin{bmatrix} x \\ 0 \\ \vdots \\ 0 \end{bmatrix} : x \in R \right\}, \quad \left\{ \begin{bmatrix} x \\ y \\ 0 \\ \vdots \\ 0 \end{bmatrix} : x, y \in R \right\}, \ldots$

Q: What are the $M_n(R)$-submodules of R^n?
$\{0\}$ trivial submodule
R^n

Homomorphisms.

For any algebraic structures A, B with the same type of operations $f, \dots$, a *homomorphism* from A to B is a function $\varphi\colon A \to B$ that preserves all operations, i.e.

$$\varphi\left(f^A(x_1, \dots, x_n)\right) = f^B\left(\varphi(x_1), \dots, \varphi(x_n)\right) \quad \text{for all } x_1, \dots, x_n \in A.$$



Definition. Let M, N be R-modules for a ring R.
$\varphi\colon M \to N$ is an *R-module homomorphism* if

(1) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$,
(2) $\varphi(rx) = r\varphi(x)$ for all $r \in R, x \in M$.

Definition. In general

- Bijective homomorphisms from A to B are called *isomorphisms.* A and B are *isomorphic* (denoted $A \cong B$) if there exists an isomorphism $A \to B$.
- Isomorphisms from A to A are called automorphisms of A.
  The set of automorphisms, $\mathrm{Aut}\, A$, is a group under composition, the automorphism group of A.
- Homomorphisms from A to A are called endomorphisms.
  The set of endomorphisms, $\mathrm{End}\, A$, is a semigroup under composition, the endomorphism monoid of E.

Example.

Let $V = R^n$ an R-module over R commutative.

1) $\mathrm{End}\, V \cong M_n(R)$

cf. Linear Algebra: choosing a basis for V yields homomorphisms $\Leftrightarrow$ $n \times n$-matrices

2) $\mathrm{Aut}\, V \cong GL_n(R)$

Outlook: Analyzing linear maps.

Example. Let V be an F-module (vector space) and $T\colon V \to V$ an F-module homomorphism (linear map).

Then V is an $F[x]$-module under

$$F[x] \times V \to V,\ (p(x), v) \mapsto \underbrace{p(T)}_{\text{linear map}}(v).$$

Note: The action of $F[x]$ on $V =: V_T$ depends on the choice of T.

Assume $\underline{V_T \cong V_S}$ for $S, T: V \to V$ linear.

Then there exists $\varphi: V_T \to V_S$ bijective s.t.

$$\varphi(T(v)) = \varphi(\underbrace{x\,v}_{\in V_T}) = x\,\underbrace{\varphi(v)}_{\in V_S} = S\,\varphi(v)$$

Hence $T(v) = \varphi^{-1} S \varphi(v)$ for all $v \in V$.

$\underline{S, T \text{ are conjugate}}$ (similar)

Converse holds as well.

Definition. For R-modules M, N, let

$$\mathrm{Hom}_R(M, N) := \{\varphi\colon M \to N \ :\ \varphi \text{ is an } R-\text{module homomorphism}\}.$$

Then $\mathrm{Hom}_R(M, N)$ is an abelian group under

$$(\varphi + \psi)(x) :=$$

Question. Is $\mathrm{Hom}_R(M, N)$ an R-module?

For an R-module M, the set of endomorphisms $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ forms a ring under $+$ and composition, called the *endomorphism ring* of M.

Question. Is $\mathrm{End}_R(M, N)$ an R-algebra?

Example. Let $V = F^n$ be an F-module for a commutative ring F.