14.9 Transcendental extensions.

Idea. Extend fields by elements that are not algebraic (i.e., are transcendental).

Definition. Let K/F.
$S \subseteq K$ is *algebraically independent over* F if for all finite subsets $\{\alpha_1, \ldots, \alpha_n\} \subseteq S$ and all $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$

$$f(\alpha_1, \ldots, \alpha_n) = 0 \;\Rightarrow\; f(x_1, \ldots, x_n) = 0.$$

$\alpha \in K$ is *transcendental over* F if $\{\alpha\}$ is algebraically independent.

A *transcendence basis* $B \subseteq K$ over F is a maximal algebraically independent subset of K.

Example. $\{t\}$ is alg independent over F in any rational function field $F(t)$.

If $\alpha \in K$ is transcendental over F, then $F(\alpha) \cong F(t)$.

E.g. $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$

Theorem. *Every K/F has a transcendence basis, and all such bases have the same cardinality.*

Proof. Zorn's Lemma & Replacement similar to Lin Algebra. □

Definition. The *transcendence degree* of K/F is the cardinality of its transcendence basis.

Example. 1) K/F is algebraic iff its transcendence basis is $\emptyset$, empty.

2) $\mathbb{Q}(t)/\mathbb{Q}$ has basis $\{t\}$, $\{t^2\}$, $\{t^2+t\}$, $\vdots$

$$\begin{array}{c} \mathbb{Q}(t) \\ | \\ \mathbb{Q}(t^2) \\ | \\ \mathbb{Q} \end{array} \quad \text{Galois (between } \mathbb{Q}(t) \text{ and } \mathbb{Q}(t^2))$$

3) $\mathbb{Q}(x_1, \ldots, x_n)/\mathbb{Q}$ has basis $\{x_1, \ldots, x_n\}$, $\{s_1, \ldots, s_n\}$, $\{p_1, \ldots, p_n\}$ ...

4) Transcendence degree of $\mathbb{R}/\mathbb{Q}$, $\mathbb{C}/\mathbb{Q}$ is 2^{ω}.

Corollary: $\exists K \subsetneq \mathbb{C}$: $K \cong \mathbb{C}$ as fields.

5) Transcendence degree $\mathbb{Q}(e, \pi) \in \{1, 2\}$, not known.

Open: Is $e + \pi \in \mathbb{Q}$?

Schanuel's Conjecture. (Transcendental Number Theory)
For any $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ that are linearly independent over $\mathbb{Q}$, the transcendence degree of

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})/\mathbb{Q}$$

is $\geq n$.

Open.

Ex Assuming Schanuel's Conjecture, for $\alpha_1 = 1$, $\alpha_2 = \pi i$:
$\mathbb{Q}(1, \pi i, e^1, \underbrace{e^{\pi i}}_{=-1}) / \mathbb{Q}$ has transcendence degree 2.

Theorem. *Let t be transcendental over F.*

1. (Lüroth) *If $F \leq K \leq F(t)$, then $K = F(s)$ for some $s \in F(t)$.*
2. *Let $p(t), q(t) \in F[t]$ be non-constant and coprime. Then*

$$[F(t) : F(p(t)/q(t))] = \max(\deg p(t), \deg q(t)).$$

Proof. □

Corollary. Aut $(F(t)/F) \cong PGL_2(F)$.

Proof. □

Theorem (Hilbert)**.** *Let $L = \mathbb{Q}(x_1, \ldots, x_n)$ with transcendence basis $x_1, \ldots, x_n$, let $G \leq$ Aut $(L/\mathbb{Q})$ be finite and*

$$K = \mathrm{Fix}(G) = \mathbb{Q}(a_1, \ldots, a_n)$$

with transcendence basis $a_1, \ldots, a_n$.

Then there are infinitely many $f\colon \{a_1, \ldots, a_n\} \to \mathbb{Q}$ whose extension $\bar{f}\colon \{x_1, \ldots, x_n\} \to \mathbb{C}$ satisfies

$$\mathrm{Gal}(\mathbb{Q}(\bar{f}(x_1), \ldots, \bar{f}(x_n))/\mathbb{Q}) \cong G.$$

Corollary. *For any $n \in \mathbb{N}$, S_n is a Galois group for some $K/\mathbb{Q}$.*

Proof. □

Infinite Galois extensions.

All our results were developed for **finite** Galois extensions. More generally we could have defined:

Definition. K/F is *Galois* if K/F is algebraic, normal and separable.

In general there is no bijection between subgroups of $\mathrm{Gal}(K/F)$ and fields $F \leq E \leq K$ for infinite extensions.

Example. $K := \mathbb{Q}(\sqrt{p}\ :\ p$ is prime) is a (countably) infinite Galois extensions of $\mathbb{Q}$.

$\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ is uniquely determined by
$\sigma(\sqrt{p}) = \pm\sqrt{p}$ for primes p.

$\sigma^2 = \mathrm{id}$ and $G := \mathrm{Gal}(K/\mathbb{Q})$ is an elementary abs 2-group
(inf dim vectorspace V over $\mathbb{F}_2$)

Recall: $V^* = \mathrm{Hom}(V, \mathbb{F}_2)$ is uncountable.

Hence V has uncountably many subspaces of codimension 1 in V.
G —"— subgroups of index 2.

But there are only countably many quadratic extension of $\mathbb{Q}$ in $\mathbb{R}$.

Rescue: use topologically closed subgroups.