13.3 Classical straightedge and compass constructions.

no measurements

Question. Using only a compass and a straightedge is it possible to

(1) construct a cube with twice the volume of a given cube?
(2) trisect any given angle?
(3) construct a square whose area is equal to that of a given circle?

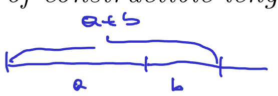
Question. More generally, given a fixed point in $\mathbb{R}^2$ and the unit length 1, which points/lengths can be constructed with compass and straightedge?

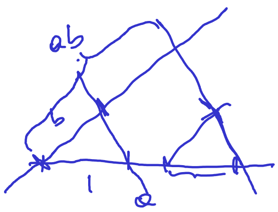
The possible construction steps are:

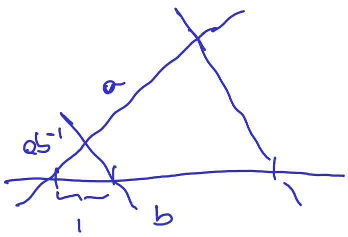
- Connect 2 points with a line.
- Circle around a point through another point.
- Intersect 2 lines.
- Intersect a line and a circle.
- Intersect 2 circles.

Lemma. *Assume $a, b \in \mathbb{R}$ are constructible. Then so are $a \pm b, ab, ab^{-1}$ (if $b \neq 0$). The set of constructible lengths C is a subfield of $\mathbb{R}$.*

Proof.

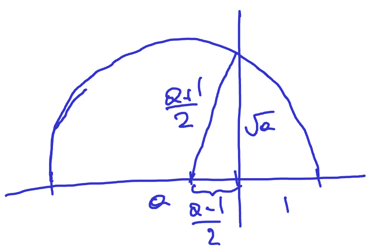






Lemma. *If $a \in C$, then $\sqrt{a} \in C$.*

Proof.



$$\sqrt{\left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2} = \sqrt{a}$$

Constructible field extensions.

Question. What extensions of a subfield F of $\mathbb{R}$ can be obtained by straightedge and compass?

1 Intersection of lines.

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

for coefficients in F.

Solutions x, y are in F.

2 Intersection of line and circle

$$\begin{aligned} ax + by &= c \\ (x-d)^2 + (y-e)^2 &= f^2 \end{aligned}$$

Solutions are roots for quadratic equations over F.

3 Intersection of circles

$$\begin{aligned} (x-a)^2 + (y-b)^2 &= c^2 \\ (x-d)^2 + (y-e)^2 &= f^2 \end{aligned}$$

Subtracting equations yields a linear equation in x, y. Hence this reduces to 2.

Lemma. *If $\alpha \in \mathbb{R}$ is obtained from a subfield F using only straightedge and compass constructions, then $[F(\alpha) : F] = 2^k$ for some $k \in \mathbb{N}$.*

Proof. Only sequences of quadratic extensions possible. □

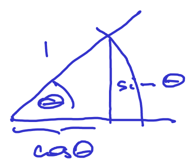
Corollary. *None of the classical Greek problems of doubling the cube, trisecting any angle, and squaring the circle are solvable with straightedge and compass.*

Proof. 1) cube with volume $V = a^3$

For a cube of volume $2a^3$ we need to construct side length $\sqrt[3]{2}\, a$.

But $m_{\sqrt[3]{2}, \mathbb{Q}}(x) = x^3 - 2$.

2) Given θ, we can construct $\sin\theta$, $\cos\theta$



Consider $\theta = \frac{\pi}{3}$

By the triple angle formula

$$\frac{1}{2} = \cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$$

For $\beta = 2\cos\frac{\theta}{3}$,

$$\beta^3 - 3\beta - 1 = 0$$

By Eisenstein's criterion $m_{\beta, \mathbb{Q}}(x) = x^3 - 3x - 1$ is irreducible.

3) $A = r^2 \pi$. We would need to construct $r\sqrt{\pi}$.

But π is transcendental over $\mathbb{Q}$ (Lindemann, 1882). □

Note. Doubling the cube and trisecting any angle is possible with a ruler (having marks for length 1) and compass.

HW, book