The dual space of an infinite dimensional vector space

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We assume the Axiom of Choice throughout. For a vector space V over field F let $\dim_F V$ denote the cardinality of a basis of V. Let $V^* := \mathrm{Hom}_F(V, F)$ denote the dual space of V.

Theorem. *Let V be an infinite dimensional F-vector space. Then $\dim_F V < \dim_F V^*$.*

For the proof we need some auxiliary results. Let B be an infinite basis of V over F.

Lemma 1. $|V| = |B| \cdot |F| = \max(|B|, |F|)$.

Proof. Easy. □

Lemma 2. $\dim_F(V^*) = |V^*|$.

Proof. By Lemma 1 applied to V^*, it suffices to show that

$$\dim_F(V^*) \geq |F|. \tag{1}$$

Assume F is infinite since (1) is trivial otherwise. Let $E := \{e_i \ : i \in \mathbb{N}\} \subseteq B$ be countably infinite. For $a \in F$ define a linear functional

$$\varphi_a \colon V \to F \text{ by } \varphi_a(e_i) := a^i \text{ for } i \in \mathbb{N} \text{ and } \varphi_a(b) := 0 \text{ for } b \in B \setminus E.$$

We claim that

$$\{\varphi_a \ : a \in F \setminus \{0\}\} \text{ is linearly independent.} \tag{2}$$

For this let $a_1, \dots, a_n \in F \setminus \{0\}$ be distinct and $c_1, \dots, c_n \in F$ be such that $\sum_{i=1}^n c_i \varphi_{a_i} = 0$. Evaluation at $e_1, \dots, e_n$ yields the linear system

$$\sum_{i=1}^{n} c_i a_i^j = 0 \quad (0 \leq j \leq n-1).$$

Recall from Linear Algebra that the $n \times n$-matrix (a_i^j) has determinant $\prod_{1 \leq i < j \leq n} (a_j - a_i) \neq 0$ (Vandermonde matrix). Hence the coefficient matrix (a_i^j) in the linear system above is invertible and $c_1 = \dots = c_n = 0$. This proves (2).

Since V^* contains a linearly independent set of size $|F|$, we have (1) and $\dim_F(V^*) = |V^*|$. □

Note that $V^* \cong F^B$ since every linear functional is uniquely determined on B (HW).

Hence by Lemma 2

$$\dim_F V^* = |F|^{|B|} > |B| = \dim_F V.$$

The strict inequality above follows from Cantor's diagonalization argument that the power set of B has cardinality strictly greater than $|B|$. This completes the proof of the Theorem. $\square$

Remark 1. *Note that V is isomorphic to the direct sum FB of B copies of F and V^* is isomorphic to the direct product F^B of B copies of F. By the Theorem above FB and F^B are not isomorphic if B is infinite.*

Remark 2. *For an infinite set I and nontrivial modules M_i $(i \in I)$, the natural embedding from the direct sum $\bigoplus_{i\in I} M_i$ into the direct product $\prod_{i\in I} M_i$ is not surjective. However there may be some other isomorphism from the direct sum to the direct product.*

As an example, for $i \in \mathbb{N}$ consider $M_i = \mathbb{R}$ as vector space over $\mathbb{Q}$. By Lemma 1, $\dim_{\mathbb{Q}} M_i = |\mathbb{R}| = c$ (continuum) and

$$\dim_{\mathbb{Q}} \bigoplus_{i\in\mathbb{N}} M_i = |\mathbb{R}| \cdot |\mathbb{N}| = c.$$

For the direct product we see that

$$|\prod_{i\in\mathbb{N}} M_i| = |\mathbb{R}|^{|\mathbb{N}|} = |2^{\mathbb{N}}|^{|\mathbb{N}|} = |2^{\mathbb{N}\times\mathbb{N}}| = |2^{\mathbb{N}}| = c.$$

In particular $c \leq \dim_{\mathbb{Q}} \mathbb{R}^{\mathbb{N}} \leq c$ and equality follows.

Since the direct sum $V := \bigoplus_{i\in\mathbb{N}} M_i$ and the direct product $W := \prod_{i\in\mathbb{N}} M_i$ have the same dimension over $\mathbb{Q}$, they are isomorphic as $\mathbb{Q}$-vector spaces.

On the other hand, $\dim_{\mathbb{R}} V$ is countable and $\dim_{\mathbb{R}} W = c$ by Lemma 2. Hence V and W are not isomorphic as $\mathbb{R}$-vector spaces.