12.3 Jordan canonical form.

Assume the characteristic polynomial of $A \in M_{n\times n}(F)$ splits in linear factors over F.

$\varphi : F^n \to F^n$
$v \mapsto Av$

Then each invariant factor $a(x)$ of A splits into prime powers (elementary divisors)

$$a(x) = (x-\lambda_1)^{\alpha_1}\dots(x-\lambda_l)^{\alpha_l},$$

and

$$V_A \cong F[x]/(x-\lambda_1)^{\alpha_1} \oplus \cdots \oplus F[x]/(x-\lambda_m)^{\alpha_m}.$$

Jordan blocks.
Consider a single summand $F[x]/(x-\lambda)^{\alpha}$.
Let $B = ((x-\lambda)^{\alpha-1}, \dots, x-\lambda, 1)$.
Modulo $(x-\lambda)^{\alpha}$ we have

$$x(x-\lambda)^i = (x-\lambda+\lambda)(x-\lambda)^i = (x-\lambda)^{i+1} + \lambda(x-\lambda)^i$$

$$= \begin{cases} \lambda(x-\lambda)^{\alpha-1} & \text{if } i=\alpha-1 \\ 1\cdot(x-\lambda)^{i+1} + \lambda(x-\lambda)^i & \text{if } i<\alpha-1 \end{cases}$$

With respect to B we have the $\alpha\times\alpha$ *Jordan block* with eigenvalue λ,

$$J_\alpha(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

$M^B_B\left(\varphi|_{F[x]/(x-\lambda)^\alpha}\right)$

A *Jordan canonical form* of $A \in M_{n\times n}(F)$ is a block diagonal matrix

$$\begin{pmatrix} J_{\alpha_1}(\lambda_1) & & & 0 \\ & J_{\alpha_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{\alpha_m}(\lambda_m) \end{pmatrix}$$

for the multiset $\{(x-\lambda_i)^{\alpha_i} \;:\; i \le m\}$ of elementary divisors of V.

Theorem. *Assume the characteristic polynomial of $A \in M_{n\times n}(F)$ splits in linear factors over F.*

Then A has a Jordan canonical form (which is unique up to permutation of Jordan blocks).

Example. Recall
$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$
has invariant factors $a_1(x) = a_2(x) = (x-1)^2$.

JCF $\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$, not $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

In general, JCF of A is ~~not~~ uniquely determined by char pol $c_A(x)$!
Need prime factorization of the invariant factors $a_1(x), \dots, a_n(x)$

Note. A *Jordan canonical form* for a linear map φ on a finite dimensional vector space V is a matrix representing φ in Jordan canonical form.

Theorem. *For $\varphi \in \mathrm{End}_F(V)$ TFAE:*

(1) *There exists a basis B of V such that $M_B^B(\varphi)$ is diagonal.*
(2) *A Jordan canonical form of φ is diagonal.*
(3) *The minimal polynomial $m_\varphi(x)$ splits in distinct linear factors over F.*

Proof. 1) $\Rightarrow$ 3) Assume $M_B^B(\varphi) = \mathrm{diag}(\lambda_1, \dots, \lambda_1, \dots, \lambda_n, \dots, \lambda_n) =: A$
for distinct $\lambda_1, \dots, \lambda_n \in F$.
Then $m_\varphi(x) = (x-\lambda_1) \cdots (x-\lambda_n)$ annihilates V_φ since
$(A - \lambda_1 I) \cdots (A - \lambda_n I) = 0$
and is the smallest such polynomial, hence is the min pol of φ.

3) $\Rightarrow$ 2) by def of JCF

2) $\Rightarrow$ 1) clear $\square$

Question. When is the rational canonical form of φ diagonal?

Invariant factors are all linear $a_1(x) = a_2(x) = \dots = a_n(x) = x - \lambda$
i.e. $\varphi(v) = \lambda v$ for some $\lambda \in F$