12.2 Canonical forms for linear maps.

Recall. Assume F^n has a basis B of eigenvectors of $\varphi \in \mathrm{End}_F(F^n)$.

- Then $M_B^B(\varphi)$ is a diagonal matrix.
- $M_C^C(\varphi)$ is diagonalizable for any basis C.

Question. What may prevent $A \in M_{n\times n}(F)$ to be diagonalizable?

(1) The *characteristic polynomial* $c_A(x) := \det(xI - A)$ does not have enough roots (*eigenvalues* of A) in F.

Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad c_A(x) = x^2+1$ has no roots in $\mathbb{R}$, only $i, -i \in \mathbb{C}$
So A cannot be diagonalized over $\mathbb{R}$, $A \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ in $\mathbb{C}$.

(2) The dimension of the *eigenspace* $\{v \in F^n \ : \ Av = \lambda v\}$ for the eigenvalue λ is too small.

Ex $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad c_B(x) = (x-2)^2$
$\dim \mathrm{Nul}(2I - B) = 1$

Goal. Block-diagonalize A.
Let V be a finite dimensional vector space over a field F and $\varphi \in \mathrm{End}_F(V)$.
Make V into an $F[x]$-module V_φ by

$$xv := \varphi(v) \text{ for } v \in V.$$

Recall: $F[x]$ is a PID.

(1) Rational canonical form of φ uses the invariant factors of V_φ.
(2) Jordan canonical form of φ uses the elementary divisors of V_φ.

Rational canonical form.

$$V_\varphi \cong F[x]/(a_1(x)) \oplus \cdots \oplus F[x]/(a_k(x)) \oplus \cancel{F[x]^r}$$

where $a_1(x) \,|\, a_2(x) \,|\, \ldots \,|\, a_k(x)$ and $r \geq 0$.

Note.

(1) $\dim_F V < \infty$ implies $r = 0$.
(2) $a_i(x)$ are uniquely determined if we require them to be monic. (leading coefficient 1)
(3) $\mathrm{Ann}_{F[x]}(V_\varphi) = (a_k(x))$;
$m_\varphi(x) := a_k(x)$ is the *minimal polynomial* of φ.
$m_\varphi(\varphi) = 0$ in $\mathrm{End}_F(V)$

$m_\varphi(x) \, v = 0 \quad \forall v \in V$

Companion matrices.
Consider a single summand for $a(x) = b_0 + b_1 x + \cdots + b_{d-1}x^{d-1} + x^d$.

$$F[x]/(a(x)) \cong F\{1, x, \ldots, x^{d-1}\}$$

as $F[x]$-modules with action

$$x \cdot x^i = \begin{cases} x^{i+1} & \text{if } i < d-1, \\ -b_0 - b_1 x - \cdots - b_{d-1}x^{d-1} & \text{if } i = d-1. \end{cases}$$

With respect to $B = (1, x, \ldots, x^{d-1})$, the *companion matrix* $C_{a(x)}$ of $a(x)$ is

$$C_{a(x)} := M_B^B(\varphi|_{F[x]/(a(x))}) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -b_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -b_{d-2} \\ 0 & \cdots & 0 & 1 & -b_{d-1} \end{pmatrix}$$

char polynomial of $C_{a(x)}$ is $a(x)$ (HW)

The *rational canonical form* of $\varphi \in \mathrm{End}_F(V)$ is the block diagonal matrix

$$\begin{pmatrix} C_{a_1(x)} & & & 0 \\ & C_{a_2(x)} & & \\ & & \ddots & \\ 0 & & & C_{a_k(x)} \end{pmatrix} \in M_{n\times n}(F)$$

for the invariant factors $a_1(x) \,|\, a_2(x) \,|\, \ldots \,|\, a_k(x)$ of V_φ.

Theorem. *Let* $\varphi \in \mathrm{End}_F(V)$. *Then*
(1) φ *has a unique rational canonical form;* $M_B^B(\varphi)$ = rational can form of φ
(2) *the rational canonical form determines* φ *up to similarity.*

Note. All concepts and results transfer to $n \times n$-matrices. via $\varphi: F^n \to F^n$, $x \mapsto Ax$

Theorem. *Let* $A \in M_{n\times n}(F)$.
(1) *The characteristic polynomial* $c_A(x)$ *is the product of all invariant factors of* A.
(2) (Cayley-Hamilton Theorem) *The minimal polynomial* $m_A(x)$ *divides* $c_A(x)$.
(3) $c_A(x) | m_A(x)^l$ *for some* $l \geq 1$.

(2) implies $c_A(A) = 0$

Converting $A \in M_{n\times n}(F)$ to rational canonical form.

Make $V := F^n$ into an $F[x]$-module V_A by $xv = Av$ for $v \in F^n$. Then

$$V_A \cong F[x]/(a_1(x)) \oplus \cdots \oplus F[x]/(a_k(x))$$

with invariant factors $a_1(x) \,|\, a_2(x) \,|\, \ldots \,|\, a_k(x)$ of A.

Note. The proof of this Structure Theorem for V_A is constructive: V_A is a homomorphic image of the free module $F[x]^n$ and has a finite presentation

$$V_A = \langle x_1, \ldots, x_n \; : \; (xI - A^T)(x_1, \ldots, x_n)^T = 0 \rangle.$$

$\varphi: F[x]^n \to V_A$, $x_i \mapsto e_i$ has kernel $\left\langle (xI - A^T)\cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle$

Transfer $xI - A^T$ (equivalently $xI - A$) into a diagonal matrix (its *Smith normal form*)

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & a_1(x) & & \\ & & & & \ddots & \\ & & & & & a_k(x) \end{pmatrix}.$$

with $a_1(x)|a_2(x)\ldots|a_k(x)$ the invariant factors of A.

For that we use the following row and column operations:

(1) swapping two rows (columns),
(2) adding a multiple of a row (column) to another,
(3) multiplying a row (column) by a unit in $F[x]$.

Thus the invariant factors of A are the non-unit entries of the Smith normal form of $xI - A$.

Example. (Dummit & Foote, p 482)

Are the following similar over $\mathbb{Q}$?

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

$c_A(x) =$ $(x-2)^2(x-3) = c_C(x)$

min polynomials are either $(x-2)(x-3)$ or $(x-2)^2(x-3)$

Example. (Dummit & Foote, p 485)

Determine the rational canonical form of

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

Simplify $xI - A$:

swap

$$\begin{pmatrix} x-1 & -2 & 4 & -4 \\ -2 & x+1 & -4 & 8 \\ -1 & 0 & x-1 & 2 \\ 0 & -1 & 2 & x-3 \end{pmatrix}$$

$\cdot 2$ $\cdot(x-1)$

$$\begin{pmatrix} 1 & 0 & -x+1 & -2 \\ -2 & x+1 & -4 & 8 \\ x-1 & -2 & 4 & -4 \\ 0 & -1 & 2 & x-3 \end{pmatrix}$$

multiplied new row 1 by (-1)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & x+1 & -2x-2 & 4 \\ x-1 & -2 & x^2-2x+5 & 2x-6 \\ 0 & -1 & 2 & x-3 \end{pmatrix}$$

$/\cdot 2 \;]+ \quad /\cdot(-x+1) \;]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x+1 & -2x-2 & 4 \\ 0 & -2 & x^2-2x+5 & 2x-6 \\ 0 & -1 & 2 & x-3 \end{pmatrix}$$

swap

$\cdot 2$ $\cdot(x-3)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & +1 & -2 & -(x-3) \\ 0 & -2 & x^2-2x+5 & 2x-6 \\ 0 & x+1 & -2x-2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & x^2-2x+1 & 0 \\ 0 & x+1 & 0 & x^2-2x+1 \end{pmatrix}$$

$\cdot 2 \;]+ \quad /\cdot(-x-1) \;]+$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x-1)^2 & 0 \\ 0 & 0 & 0 & (x-1)^2 \end{pmatrix}$$

Smith normal form of A

Invariant factors: $a_1(x) = a_2(x) = (x-1)^2 = m_A(x)$

$V_A \cong \mathbb{Q}[x]/(a_1(x)) \oplus \mathbb{R}[x]/(a_2(x))$

$C_{a_i(x)} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ for $i = 1, 2$

The rational canonical form of A is

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$