More NP-complete problems

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Cliques

- $K_n = (\{1, \dots, n\}, \neq)$ ... **complete (undirected) graph** on n-vertices
- A graph G has an n-**clique** if K_n embeds into G.
- CLIQUE := $\{(G, n) \ : \ G$ is a graph with n-clique$\}$

Theorem
CLIQUE is NP-complete.

Proof.
CLIQUE $\in$ NP since a guessed n-clique can be verified in polynomial time.

Claim: 3-SAT $\leq_m^p$ CLIQUE

Given a 3-SAT instance

$$\Phi = (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_n \vee b_n \vee c_n)$$

with literals a_i, b_i, c_i.

For the reduction construct a graph G with

- $3n$ vertices labelled $a_1, b_1, c_1, \ldots, a_n, b_n, c_n$
- edges between any 2 vertices except within any triple a_i, b_i, c_i representing a clause of Φ and between a variable and its negation.

Φ with $3n$ literals yields a graph with $3n$ vertices in polytime.

Example: $\Phi = (x_1 \vee x_1 \vee x_2) \wedge (x_1' \vee x_2' \vee x_2') \wedge (x_1' \vee x_2 \vee x_2)$

Claim: Φ is satisfiable iff G has an n-clique

$\Rightarrow$:

- Assume Φ has a satisfying assignment.
- In each triple a_i, b_i, c_i of G choose a vertex corresponding to a true literal in this satisfying assignment.
- These n vertices are each pairwise connected, hence a clique.

$\Leftarrow$:

- Assume G has an n-clique.
- Then any 2 vertices in that clique are in distinct clauses.
- Assign truth values to variables in Φ such that each literal labelling a vertex in the clique is true (possible since x_j, x'_j are not connected).
- Since each clause contains a vertex from the clique, this assignment satisfies Φ.

$\square$

Graph coloring

- A graph G is n-**colorable** if its vertices can be colored in n colors such that any adjacent vertices have distinct colors.
- Equivalently, G is n-colorable iff there exists a homomorphisms $G \to K_n$.
- n-Coloring $:= \{G \;:\; G \text{ is } n\text{-colorable}\}$.

Theorem

3-Coloring is NP-complete.

Proof.

3-Coloring $\in$ NP since a guessed coloring can be verified in polynomial time.

Claim: 3-SAT $\leq_m^p$ 3-Coloring

Given a 3-SAT instance

$$\Phi = (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_n \vee b_n \vee c_n)$$

with literals a_i, b_i, c_i.

Construct G that is 3-colorable iff Φ is satisfiable as follows:

- Truth assignments of $x_1, \ldots, x_k$ correspond to colors $0, 1$ of vertices.

- For each clause $a_i \vee b_i \vee c_i$ connect the vertices corresponding to a_i, b_i, c_i by a **gadget graph** implementing "or".

Further NP-complete problems (Karp, 1972)

- SAT
- Circuit Satisfiability Problem
  Given a Boolean circuit, is there an assignment of inputs that yields output 1?
- CLIQUE
- Graph k-Coloring for $k \geq 3$
- Graph Homomorphism Problem
  Given graphs G, H, is there a homomorphism $G \to H$?
- Hamiltonian Cycle for digraphs
- Travelling Salesman Problem

- Exact Cover
  Given subsets $A_1, \dots, A_n \subseteq X$, is X the disjoint union of some A_i?
- Knapsack (Subset Sum)
  Given integers $a_1, \dots, a_n$ and $s \in \mathbb{Z}$, does a non-empty subset of the a_i sum to s?
- MaxCut
  Given a graph G and $k \in \mathbb{N}$, is there a cut of size at least k in G (a partition of vertices into 2 sets A, B with $\geq k$ edges between A and B)?
- Sudoku for n^2 numbers