The Immerman-Szelepcsényi Theorem

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Note

- The complement of a complexity class C is denoted co-C.
- Clearly deterministic complexity classes are closed under complements but is not known whether NP = co-NP.
- The following was proved independently by Immerman and Szelepcsényi (then an undergrad) in 1987 and won them the Gödel prize in 1995.

Immerman-Szelepcsényi Theorem

$\mathrm{NSPACE}(s(n)) = \text{co-NSPACE}(s(n))$ for $s(n) \geq \log n$.

Corollary

NL=co-NL

Proof.

Idea: Suppose we have a non-deterministic membership test for some $A \in \Sigma^n$ and we know $|A|$.
Then we also have a non-deterministic membership test for $\Sigma^n \setminus A$:

- on input x, successively guess $|A|$ distinct strings $\neq x$ and show they are in A.

Let M be a nondeterministic TM with input and work tape that runs in $s(n)$-space.

- For fixed input of length n, every configuration of M (state, positions of tape heads, content of work tape) can be represented by an $s(n)$-tuple over some finite C (uses $s(n) \geq \log n$ to encode the position $\leq n$ on input tape).
- Let $\mathtt{start}, \mathtt{accept} \in C^{s(n)}$ be the unique starting and accepting configuration on inputs of length n.
- For $m \in \mathbb{N}$ let A_m be the set of configurations that can be reached from $\mathtt{start}$ in $\leq m$ steps.
- Then M accepts x iff $\mathtt{accept} \in A_{|C|^{s(n)}}$.

A non-deterministic TM N with L(N) = $\overline{L(\text{M})}$

On input x of length n, compute $|A_0|, |A_1|, \ldots, |A_{|C|^{s(n)}}|$ inductively:

1. $|A_0| := 1$
2. Given $|A_m|$, set $|A_{m+1}| := 0$ and enumerate $\beta \in C^{s(n)}$ lexicographically.
3. To check $\beta \in A_{m+1}$,
   - 3.1 non-deterministically guess the $|A_m|$ elements $\alpha \in A_m$ in lexicographical order,
   - 3.2 guess and verify a path of length $\leq m$ from `start` to α and
   - 3.3 check that β is a successor of α.
4. If $\beta \in A_{m+1}$, then increase the counter for $|A_{m+1}|$ by 1. Repeat for the next β in lexicographical order.

For $m = |C|^{s(n)}$ enumerate $\alpha \in A_{|C|^{s(n)}}$ in lexicographical order as in steps 3.1-2 above.

If `accept` $\neq \alpha$ for all of them, then $x \notin L(\text{M})$ and N accepts x.

Space complexity of N:

- At each step N's working tape holds m, $|A_m|, |A_{m+1}|$, β, α, i and 2 intermediate configurations in A_i, A_{i+1} on a path from `start` to α.
- Each of these takes $s(n)$ space.

Hence N runs in $O(s(n))$ space.

Recall:
To avoid the assumption that $s(n)$ is space constructible,
let N run for successive values $s = \log n, \log n + 1, \ldots$

- Whenever M tries to use more than s space, restart N with $s + 1$ space.
- Since M runs in $s(n)$ space, it eventually will not reach any longer configurations and the loop above stops with N only ever using $O(s(n))$ space.

$\square$