Incomparable degrees

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So far we constructed only degrees $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \ldots$

Question

- Are there any other Turing degrees?
- Are Turing degrees linearly ordered?

Note

- By the Jump Theorem 4, the jump is monotonous, i.e.

$$\mathbf{b} \leq \mathbf{a} \;\Rightarrow\; \mathbf{b}' \leq \mathbf{a}'.$$

- Hence $\mathbf{0}' \leq \mathbf{a}'$ for every degree $\mathbf{a}$.

The jump is not injective

Theorem (Spector, 1956)

There exists a degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{a}' = \mathbf{0}'$.

Proof.

We want to construct $A \subseteq \mathbb{N}$ such that

- A is not computable, $R_{2e}: \ \chi_A \neq \varphi_e$.
- $A' \leq \emptyset'$ $R_{2e+1}: \ \varphi_e^A(e)\downarrow$ is $\emptyset'$-computable.

To satisfy these requirements for all $e \in \mathbb{N}$ define finite initial segments σ_s of χ_A in stages enumerated by $s \in \mathbb{N}$.

Initialize $\sigma_0 := ()$.
Stage $s = 2e$ [Put x into A iff $\varphi_e(x) = 0$]:
Given σ_s, let $x := |\sigma_s|$ and test $\emptyset'$-computably whether

$$\exists t : \ \varphi_{e,t}(x)\downarrow \text{ and } \varphi_{e,t}(x) = 0.$$

If yes, let $\sigma_{s+1} := \sigma_s \circ (1)$; else $\sigma_{s+1} := \sigma_s \circ (0)$.

Stage $s = 2e + 1$ [Decide whether $e \in A'$]:
Given σ_s test $\emptyset'$-computably whether

$$\exists \sigma \ \exists t : \ \sigma_s \subseteq \sigma, \ \varphi^{\sigma}_{e,t}(e) \downarrow$$

Let σ_{s+1} be the smallest such σ if it exists; else $\sigma_{s+1} := \sigma_s$.

Let $A \subseteq \mathbb{N}$ with characteristic function $\bigcup_{s \in \mathbb{N}} \sigma_s$. Then

1. $A >_T \emptyset$ since it satisfies R_{2e} for all e;
2. σ_s and hence A is computable in $\emptyset'$ by construction;
3. $A' \leq_T \emptyset'$ since
$$e \in A' \text{ iff } \varphi_e^{\sigma_{2e+2}}(e) \downarrow$$
where σ_{2e+2} is $\emptyset'$-computable by 2.
4. $A' \geq_T \emptyset'$ by the previous remark. □

Note

A from the proof is $\leq \emptyset'$, hence Δ_2 by Post's Theorem (not necessarily Σ_1).

The jump is onto degrees $\geq$ 0'

Friedberg Completeness Criterion

For every degree $\mathbf{b} \geq \mathbf{0}$', there is $\mathbf{a}$ such that $\mathbf{b} = \mathbf{a} \vee \mathbf{0}$' $= \mathbf{a}$'.

Proof

Let $B \subseteq \mathbb{N}$ with $\emptyset' \leq_T B$. Construct A such that

- $B \leq_T A \oplus \emptyset'$
- $A' \leq_T B$,

via finite initial segments σ_s $(s \in \mathbb{N})$ of χ_A.

Initialize $\sigma_0 := ()$.

Stage $s = 2e$ [Code $\chi_B(e)$ into A]: $\sigma_{s+1} := \sigma_s \circ (\chi_B(e))$
Stage $s = 2e + 1$ [Decide whether $e \in A'$, cf. Spector's Theorem]:
Given σ_s test $\emptyset'$-computably whether

$$\exists \sigma \ \exists t : \ \sigma_s \subseteq \sigma, \ \varphi^{\sigma}_{e,t}(e) \downarrow$$

Let σ_{s+1} be the smallest such σ if it exists; else $\sigma_{s+1} := \sigma_s$

Let $A \subseteq \mathbb{N}$ with characteristic function $\bigcup_{s \in \mathbb{N}} \sigma_s$.

1. σ_s and hence A is computable in $B \geq_T \emptyset'$ by construction.
2. $A' \leq_T B$ since
$$e \in A' \text{ iff } \varphi_e^{\sigma_{2e+2}}(e) \downarrow$$
where σ_{2e+2} is B-computable by 1.
3. σ_s is computable in $A \oplus \emptyset'$ by induction on s:
    - Given σ_{2e}, compute $\sigma_{2e+1} = \sigma_{2e} \circ (\chi_A(|\sigma_{2e}|))$ with A-oracle.
    - Given σ_{2e+1}, compute σ_{2e+2} using $\emptyset'$-oracle.
4. $B \leq_T A \oplus \emptyset'$ by 3. since $\chi_B(e)$ is the last entry in σ_{2e+1}. $\square$

Incomparable degrees exist

Theorem (Avoiding cones)

For every degree $\mathbf{b} > \mathbf{0}$ there exists $\mathbf{0} < \mathbf{a} < \mathbf{b}$' such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Proof.

Let B be non-computable. Construct A such that

- A is not computable $\qquad R_{2e}: \ \chi_A \neq \varphi_e$
- Whenever $C \leq_T A$ and $C \leq_T B$, then C is computable

  $R_{2\sharp(e,f)+1}: \ \chi_C = \varphi_e^A = \varphi_f^B \ \Rightarrow C$ is computable.

  Here $\sharp$ denotes a computable bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Define initial segments σ_s of χ_A in stages.

Initialize $\sigma_0 := ()$.
Stage $s = 2e$**:** Put $x := |\sigma_s|$ into A iff $\varphi_e(x) = 0$.

$$\sigma_{s+1} := \begin{cases} \sigma_s \circ (1) & \text{if } \varphi_e(x)\downarrow \text{ and } \varphi_e(x) = 0, \\ \sigma_s \circ (0) & \text{else.} \end{cases}$$

Stage $s = 2\sharp(e,f)+1$**:**
Given σ_s test $\emptyset'$-computably whether

$$\exists\sigma\ \exists\tau\ \exists x:\ \sigma_s \subseteq \sigma,\tau,\ \varphi_e^{\sigma}(x)\downarrow \neq \varphi_e^{\tau}(x)\downarrow \qquad (\dagger)$$

Case 1, $(\dagger)$ holds [σ,τ are e-**splitting** extensions of σ_s]:

- Check B'-computably whether $\varphi_f^B(x) \neq \varphi_e^{\sigma}(x)$.
  If yes, set $\sigma_{s+1} := \sigma$; if no, $\sigma_{s+1} := \tau$.
- Then R_s holds since $\varphi_f^B \neq \varphi_e^A$.

Case 2, $(\dagger)$ does not hold: Set $\sigma_{s+1} := \sigma_s$.

- **Claim:** If $\chi_A \supseteq \sigma_s$ and $\psi := \varphi_e^A$ is total, then ψ is computable.
- To compute $\psi(x)$, dovetail $\varphi_e^{\tau}(x)$ for all strings extending σ_s.
- The first converging computation yields $\psi(x)$ (Since $(\dagger)$ does not hold, all converging computations yield the same result).
- Then R_s holds since $\chi_C = \varphi_e^A \Rightarrow C$ computable.

Finally

1. $\deg(A) > \mathbf{0}$ by requirements R_e,
2. $\deg(A) \wedge \deg(B) = \mathbf{0}$ by requirements $R_{2\sharp(e,f)+1}$,
3. $\deg(A) \leq \deg(B)'$ since A is computable in $\emptyset' \oplus B' \equiv_T B'$ by construction.

$\square$

Corollary (Kleene, Post)

There exist incomparable degrees $\leq \mathbf{0}$'.

Proof.

- By Spector's Theorem we have $\emptyset <_T B <_T \emptyset'$.
- By the Avoiding Cones Theorem we have $A < \emptyset'$ such that

$$\deg(A) \wedge \deg(B) = \mathbf{0}, \ \deg(A) \vee \deg(B) = \mathbf{0}'.$$

- By Post's Theorem these A, B are Δ_2, not Σ_1.

$\square$