Turing degrees

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Turing reductions

A, B denote subsets of $\mathbb{N}$ throughout.

Definition

B is **Turing reducible** to A (denoted $B \leq_T A$) if B is computable in A.

Many-one vs Turing reduction

- $B \leq_m A$: $\exists$ computable f such that $\chi_B = \chi_A \circ f$
  Oracle A is queried only once after encoding x as $f(x)$.
- $B \leq_T A$: χ_B is recursive in χ_A
  Oracle χ_A can be used several times in defining χ_B.

Turing reductions are the most general computable reductions from B to A.

Definition

B is **computably enumerable in** A if $B = W_e^A$ for some e where $W_e^A = \mathrm{domain}\varphi_e^A$.

Our current theory for c.e. sets relativizes to A-c.e. sets.

Relativized Complementation Theorem

$B \leq_T A$ iff B and $\bar{B}$ are c.e. in A.

Theorem

TFAE:

1. B is c.e. in A;
2. $B = \emptyset$ or B is the range of some A-computable total function;
3. B is Σ_1^A.

Turing degrees

Definition

- A is **Turing equivalent** to B (denoted $A \equiv_T B$) if $A \leq_T B$ and $B \leq_T A$.
- $\equiv_T$ is an equivalence relation on the subsets of $\mathbb{N}$.
- The **Turing degree** of A is the equivalence class

$$\deg(A) := \{B \ : \ B \equiv_T A\}$$

- A degree is c.e. if it contains a c.e. set.
- The set of all degrees D is partially ordered by

$$\deg(A) \leq \deg(B) \text{ iff } A \leq_T B.$$

Basic order properties

1. The smallest degree in D is

$$\mathbf{0} := \deg(\emptyset) = \{A \ : \ A \text{ is computable}\}$$

2. Any two degrees have a **supremum** (least upper bound)

$$\deg(A) \vee \deg(B) = \deg(A \oplus B)$$

where

$$A \oplus B = \{2x \ : \ x \in A\} \cup \{2x+1 \ : \ x \in B\}$$

encodes the disjoint union of A and B.
(HW)

3. The supremum of infinitely many degrees and the **infimum** (greatest lower bound) of two degrees need not always exist. (postponed)

4. D is a $\vee$-semilattice, not a lattice.

Turing jump

Problem

Given a degree $\mathbf{a} = \deg(A)$ find a degree $\mathbf{b} = \deg(B)$ that is strictly greater than $\mathbf{a}$ (denoted $\mathbf{a} < \mathbf{b}$).
In other words, find B that is not A-computable.

Definition

For $A \subseteq \mathbb{N}$, the relativized diagonal Halting Problem

$$A' := K^A = \{x \ : \ \varphi_x^A(x)\downarrow\} = \{x \ : \ x \in W_x^A\}$$

is the **jump** of A.

Basic properties of the jump

Jump Theorem

For $A, B \subseteq \mathbb{N}$

1. A' is A-c.e.;
2. B is A-c.e. iff $B \leq_m A'$;
3. $A <_T A'$;
4. $B \leq_T A$ iff $B' \leq_m A'$;
5. If $B \equiv_T A$, then $B' \equiv_m A'$.

Proof 1.

By the Relativized Enumeration Theorem we have z such that

$$\forall e, x: \ \varphi_e^A(x) = \varphi_z^A(e, x).$$

Then $\psi(x) := \varphi_z^A(x, x)$ is A-computable and $A' = \mathrm{domain}\, \psi$ is A-c.e. $\square$

Proof 2.

$\Rightarrow$: Let $B = \mathrm{domain}\, \varphi_e^A$ for some e. By the Relativized S_m^n-Theorem we have a computable s such that

$$\forall x, y: \ \varphi_e^A(x) = \varphi_{s(x)}^A(y).$$

$$\begin{aligned} \text{Then } x \in B &\text{ iff } \varphi_e^A(x)\downarrow \\ &\text{ iff } \varphi_{s(x)}^A(s(x))\downarrow \\ &\text{ iff } s(x) \in A'. \end{aligned}$$

Hence s is a many-one reduction from B to A'.

$\Leftarrow$: Let s be a many-one reduction from B to A'.

$$\begin{aligned} \text{Then } x \in B &\text{ iff } \varphi_{s(x)}^A(s(x))\downarrow \\ &\text{ iff } \varphi_e^A(x)\downarrow \qquad \text{(where } e \in \mathbb{N} \text{ is determined by } s\text{)} \end{aligned}$$

Hence $B = \mathrm{domain}\, \varphi_e^A$ is A-c.e. $\square$

Proof 3, 4.

(HW)

Jump of degrees

Definition

For the degree **a** the jump

$$\mathbf{a'} := \deg(A') \text{ for some } A \in \mathbf{a}$$

is welldefined by the Jump Theorem 5.

Hence there is an infinitely ascending chain of degrees:

$\mathbf{0} = \deg(\emptyset) = \{A \ : \ A \text{ is computable } \}$
$\mathbf{0}' = \deg(\emptyset') = \deg(K)$
$\mathbf{0}'' = \deg(\emptyset'') = \deg(F) = \deg(T)$

Outlook

Question

- **Post's Problem:** Is there a c.e. degree other than $\mathbf{0}$ and $\mathbf{0}'$?
- Are there incomparable degrees?
- Does the infimum exists for any two degrees?