## Simple sets

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We are still working towards the answer to the following:

### Question

Is every non-computable c.e. set many-one complete for  $\Sigma_1^0$ ?

# Witnesses for $\Sigma_1^0$ -predicates can be computed

## Theorem ( $\Sigma_1^0$ -Uniformization Theorem)

Let  $P \subseteq \mathbb{N}^{k+1}$  be  $\Sigma^0_1$ . Then there exists a computable  $\psi \colon \mathbb{N}^k \to_p \mathbb{N}$  (a selector function for P) such that

$$\psi(\bar{x})\downarrow \text{ iff } \exists y \ P(\bar{x},y).$$

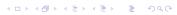
Further if  $\psi(\bar{x}) \downarrow$ , then  $P(\bar{x}, \psi(\bar{x}))$ .

#### Proof.

**b** By assumption we have computable  $R(\bar{x}, y, z)$  such that

$$P(\bar{x}, y) \equiv \exists z \ R(\bar{x}, y, z).$$

- ▶ Let  $\theta(\bar{x}) := \mu u \ R(\bar{x}, (u)_0, (u)_1)$ .
- ▶ Then  $\psi(\bar{x}) := (\theta(\bar{x}))_0$  satisfies the assertions.



## Simple sets

#### Definition

A c.e. set A is **simple** if  $\bar{A}$  is infinite and does not contain an infinite c.e. subset.

#### Note

 $simple \Rightarrow not computable$ 

## Theorem (Post, 1944)

Simple sets exist.

#### Proof.

Idea: Construct A that intersects every infinite c.e. set  $W_e$  with complement  $\bar{A}$  still infinite.

Define

$$P(e,x) \equiv x > 2e \land x \in W_e$$

with selector function

$$\psi(e) \downarrow \text{ iff } \exists x \ P(e,x)$$

- ▶ If  $W_e$  is infinite, then  $\psi(e) \downarrow$  and  $\psi(e) > 2e, \psi(e) \in W_e$ .
- ▶ Claim:  $A := \psi(\mathbb{N})$  is simple because
- ▶ 1.  $A \cap W_e \neq \emptyset$  for all infinite  $W_e$ ;
- ▶ 2.  $\bar{A}$  is infinite as  $|A \cap \{0, 1, ..., 2x\}| \le x$ .



## Creative $\Rightarrow$ not simple

Recall: productive  $\Rightarrow$  not c.e.

#### **Theorem**

Every productive set *B* contains an infinite c.e. subset.

### Proof.

Let B be productive and let p be computable such that

$$\forall x \ W_x \subseteq B \Rightarrow p(x) \in B \setminus W_x.$$

**Claim:**  $\exists$  computable f such that  $W_{f(x)} = W_x \cup \{p(x)\}.$ 

- ► Consider  $g(x,y) := \begin{cases} 0 & \text{if } y = p(x), \\ \varphi_x(y) & \text{else.} \end{cases}$
- ▶ By the  $S_n^m$ -Theorem we have a computable f such that  $\varphi_{f(x)}(y) = g(x, y)$ .

Let e with  $W_e = \emptyset \subseteq B$ . Then

$$\qquad \qquad W_{f^i(e)} = \{p(e), pf(e) \dots, pf^{i-1}(e)\} \subseteq B$$

▶ 
$$pf^i(e) \notin W_{f^i(e)}$$
 for all  $i \in \mathbb{N}$ .

Hence

$$A := \{ \underbrace{p(e)}_{e, pf(e)}, pf(e), pf^{2}(e), \dots \} \subseteq B$$

is infinite c.e.

## No computable/complete dichotomy for c.e. sets

### Corollary

There exist c.e. sets that are neither computable nor  $\Sigma_1^0$ -complete.

### Proof.

Let A be simple  $(\exists by Post's Theorem)$ . Then

- 1. A is not computable since  $\bar{A}$  is not c.e. by definition,
- 2. A is not creative ( $=\Sigma_1^0$ -complete) because  $\bar{A}$  is not productive by the previous Theorem.



## Lattice theoretic properties

#### Definition

Let  $\mathcal{E}$  denote the lattice of c.e. subsets of  $\mathbb{N}$  (under  $\cap, \cup, \emptyset, \mathbb{N}$ ). A property of c.e. sets is **lattice theoretic** if it is first order definable in  $\mathcal{E}$ .

### Examples of lattice theoretic properties

- 1. A is computable iff A has a complement in  $\mathcal{E}$  iff  $\exists B: A \cup B = \mathbb{N}, A \cap B = \emptyset$ .
- 2. A is finite iff  $\forall B : A \cap B$  is computable.
- 3. A is simple iff A is not computable and  $\forall B$  infinite:  $A \cap B \neq \emptyset$ .
- 4. A, B are **computably inseparable** iff  $\neg(\exists \text{ computable } C: A \subseteq C \text{ and } C \cap B = \emptyset)$ .
- 5. (Harrington) A is creative iff  $\exists C \supseteq A \forall B \subseteq C \exists$  computable R:  $R \cap C$  is non-computable,  $R \cap A = R \cap B$ .



### Friedberg Splitting Theorem

Let  $A \subseteq \mathbb{N}$  be c.e., noncomputable. Then A is the disjoint union of computably inseparable c.e. sets  $B_0, B_1$ .