Simple sets

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We are still working towards the answer to the following:

Question

Is every non-computable c.e. set many-one complete for Σ^0_1?

Witnesses for Σ^0_1-predicates can be computed

Theorem (Σ^0_1-Uniformization Theorem)

Let $P \subseteq \mathbb{N}^{k+1}$ be Σ^0_1. Then there exists a computable $\psi \colon \mathbb{N}^k \to_p \mathbb{N}$ (a **selector function** for P) such that

$$\psi(\bar{x}) \downarrow \text{ iff } \exists y \; P(\bar{x}, y).$$

Further if $\psi(\bar{x}) \downarrow$, then $P(\bar{x}, \psi(\bar{x}))$.

Proof.

- By assumption we have computable $R(\bar{x}, y, z)$ such that

$$P(\bar{x}, y) \; \equiv \; \exists z \; R(\bar{x}, y, z).$$

- Let $\theta(\bar{x}) := \mu u \; R(\bar{x}, (u)_0, (u)_1)$.
- Then $\psi(\bar{x}) := (\theta(\bar{x}))_0$ satisfies the assertions.

$\square$

Simple sets

Definition

A c.e. set A is **simple** if $\bar{A}$ is infinite and does not contain an infinite c.e. subset.

Note

simple $\Rightarrow$ not computable

Theorem (Post, 1944)

Simple sets exist.

Proof.

Idea: Construct A that intersects every infinite c.e. set W_e with complement $\bar{A}$ still infinite.

- Define

$$P(e, x) \equiv x > 2e \wedge x \in W_e$$

with selector function

$$\psi(e)\downarrow \text{ iff } \exists x \; P(e, x)$$

- If W_e is infinite, then $\psi(e)\downarrow$ and $\psi(e) > 2e, \psi(e) \in W_e$.
- **Claim:** $A := \psi(\mathbb{N})$ is simple because
- 1. $A \cap W_e \neq \emptyset$ for all infinite W_e;
- 2. $\bar{A}$ is infinite as $|A \cap \{0, 1, \ldots, 2x\}| \leq x$.

□

Creative $\Rightarrow$ not simple

Recall: productive $\Rightarrow$ not c.e.

Theorem

Every productive set B contains an infinite c.e. subset.

Proof.

Let B be productive and let p be computable such that

$$\forall x \; W_x \subseteq B \Rightarrow p(x) \in B \setminus W_x.$$

Claim: $\exists$ computable f such that $W_{f(x)} = W_x \cup \{p(x)\}$.

- Consider $g(x,y) := \begin{cases} 0 & \text{if } y = p(x), \\ \varphi_x(y) & \text{else.} \end{cases}$
- By the S^m_n-Theorem we have a computable f such that $\varphi_{f(x)}(y) = g(x,y)$.

Let e with $W_e = \emptyset \subseteq B$. Then

- $W_{f^i(e)} = \{p(e), pf(e) \ldots, pf^{i-1}(e)\} \subseteq B$
- $pf^i(e) \notin W_{f^i(e)}$ for all $i \in \mathbb{N}$.

Hence

$$A := \{ \underbrace{\underbrace{p(e)}_{\in B}}_{= W_{f(e)}}, pf(e), pf^2(e), \ldots \} \subseteq B$$

is infinite c.e. $\square$

No computable/complete dichotomy for c.e. sets

Corollary

There exist c.e. sets that are neither computable nor Σ^0_1-complete.

Proof.

Let A be simple ($\exists$ by Post's Theorem). Then

1. A is not computable since $\bar{A}$ is not c.e. by definition,
2. A is not creative ($=\Sigma^0_1$-complete) because $\bar{A}$ is not productive by the previous Theorem.

$\square$

Lattice theoretic properties

Definition

Let $\mathcal{E}$ denote the lattice of c.e. subsets of $\mathbb{N}$ (under $\cap, \cup, \emptyset, \mathbb{N}$).
A property of c.e. sets is **lattice theoretic** if it is first order definable in $\mathcal{E}$.

Examples of lattice theoretic properties

1. A is computable iff A has a complement in $\mathcal{E}$
   iff $\exists B: \ A \cup B = \mathbb{N}, A \cap B = \emptyset$.
2. A is finite iff $\forall B: \ A \cap B$ is computable.
3. A is simple iff A is not computable and $\forall B$ infinite: $A \cap B \neq \emptyset$.
4. A, B are **computably inseparable** iff $\neg(\exists$ computable C: $A \subseteq C$ and $C \cap B = \emptyset)$.
5. (Harrington) A is creative iff $\exists C \supseteq A \forall B \subseteq C \exists$ computable R: $R \cap C$ is non-computable, $R \cap A = R \cap B$.

Friedberg Splitting Theorem

Let $A \subseteq \mathbb{N}$ be c.e., noncomputable. Then A is the disjoint union of computably inseparable c.e. sets B_0, B_1.