Many-one completeness for arithmetical hierarchy

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What are the hardest Σ^0_n-problems?

To simplify notation we only consider subsets of $\mathbb{N}$.

Recall

- For $A, B \subseteq \mathbb{N}$, A is **many-one reducible** to B (short $A \leq_m B$) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$:

$$\forall x \in \mathbb{N} : x \in A \text{ iff } f(x) \in B.$$

- A is c.e. iff $A \leq_m AP$ (HW 4). Hence the acceptance problem is "hardest" among Σ^0_1-sets.

Question

Can this be generalized to higher levels of the arithmetical hierarchy?

Closure under many-one reductions

Lemma

If $A \leq_m B$ and B is Σ^0_n, Π^0_n, respectively, then A is Σ^0_n, Π^0_n, respectively.

Proof.

Assume $f : A \to B$ is a many-one reduction and $B(z)$ is Σ^0_n. Then

$$A(x) \equiv B(f(x))$$

is Σ^0_n since Σ^0_n is closed under substitution by total computable functions. $\square$

Σ_n^0-complete sets

Definition

$C \subseteq \mathbb{N}$ is Σ_n^0-complete if

1. C is Σ_n^0 and
2. for every Σ_n^0-set A: $A \leq_m C$.

Theorem

For each $n \geq 1$

1. Σ_n^0-complete sets exist;
2. no Σ_n^0-complete set is Π_n^0.

Universal $\Rightarrow$ complete

Proof.
1. Let $U_n(e,x)$ be a universal Σ^0_n-predicate. Then

$$C_n := \{2^e 3^x \ : \ U_n(e,x)\}$$

is Σ^0_n-complete since for each Σ^0_n A, we have $e \in \mathbb{N}$:

$$A(x) \text{ iff } U_n(e,x) \text{ iff } 2^e 3^x \in C_n.$$

2. Recall: $K_n(x) = U_n(x,x)$ is Σ^0_n, not Π^0_n.
Let C be Σ^0_n-complete.
Then $K_n \leq_m C$ and C cannot be Π^0_n either. $\square$

Further complete examples 1

$T = \{e : \varphi_e \text{ is total}\}$ is Π^0_2-complete.

Proof.

Recall T is Π^0_2. Let R be computable and

$$A(x) \equiv \forall y \exists z\ R(x,y,z) \qquad (\Pi^0_2)$$

- Define $\psi(x,y) := \mu z\ R(x,y,z)$.
- By the S^m_n-Theorem for $m = n = 1$, we have a computable $h := S^1_1$ such that

$$\psi(x,y) = \varphi_{h(x)}(y) \text{ for all } x, y.$$

- Then $x \in A$ iff $\forall y\ \varphi_{h(x)}(y) \downarrow$
  iff $\varphi_{h(x)}$ is total
  iff $h(x) \in T$.
- Hence the S^m_n-Theorem yields a many-one reduction h from A to T.

Further complete examples 2, 3

The **diagonal halting problem** $K = \{x \ : \ \varphi_x(x)\downarrow\}$ is Σ^0_1-complete.

Proof

Let R be computable and

$$A(x) \equiv \exists y \ R(x, y) \qquad\qquad (\Sigma^0_1)$$

- Define $\psi(x, z) := \mu y \ R(x, y)$ (independent of z!).
- By the S^m_n-Theorem, we have a computable h such that

$$\psi(x, z) = \varphi_{h(x)}(z) \text{ for all } x, z.$$

- Then $x \in A$ iff $\psi(x, z)\downarrow$
  iff $\varphi_{h(x)}(h(x))\downarrow$
  iff $h(x) \in K$. $\square$

$K_n := \{x \ : \ U_n(x, x)\}$ is Σ^0_n-complete for $n \geq 1$.