Many-one completeness for arithmetical hierarchy

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What are the hardest $\Sigma^0_n$-problems?

To simplify notation we only consider subsets of $\mathbb{N}$.

Recall

- For $A, B \subseteq \mathbb{N}$, $A$ is **many-one reducible** to $B$ (short $A \leq^m B$) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$:

  $$\forall x \in \mathbb{N} : x \in A \iff f(x) \in B.$$

- $A$ is c.e. iff $A \leq^m AP$ (HW 4). Hence the acceptance problem is “hardest” among $\Sigma^0_1$-sets.

**Question**

Can this be generalized to higher levels of the arithmetical hierarchy?
Closure under many-one reductions

Lemma
If $A \leq_m B$ and $B$ is $\Sigma^0_n, \Pi^0_n$, respectively, then $A$ is $\Sigma^0_n, \Pi^0_n$, respectively.

Proof.
Assume $f : A \to B$ is a many-one reduction and $B(z)$ is $\Sigma^0_n$. Then

$$A(x) \equiv B(f(x))$$

is $\Sigma^0_n$ since $\Sigma^0_n$ is closed under substitution by total computable functions.

\qed
\[\Sigma^0_n\]-complete sets

**Definition**

\(C \subseteq \mathbb{N}\) is \(\Sigma^0_n\)-complete if

1. \(C\) is \(\Sigma^0_n\) and
2. for every \(\Sigma^0_n\)-set \(A\): \(A \leq_m C\).

**Theorem**

For each \(n \geq 1\)

1. \(\Sigma^0_n\)-complete sets exist;
2. no \(\Sigma^0_n\)-complete set is \(\Pi^0_n\).
Universal $\Rightarrow$ complete

Proof.
1. Let $U_n(e, x)$ be a universal $\Sigma^0_n$-predicate. Then

$$C_n := \{2^e3^x : U_n(e, x)\}$$

is $\Sigma^0_n$-complete since for each $\Sigma^0_n A$, we have $e \in \mathbb{N}$:

$$A(x) \text{ iff } U_n(e, x) \text{ iff } 2^e3^x \in C_n.$$

2. Recall: $K_n(x) = U_n(x, x)$ is $\Sigma^0_n$, not $\Pi^0_n$.
Let $C$ be $\Sigma^0_n$-complete.
Then $K_n \leq_m C$ and $C$ cannot be $\Pi^0_n$ either. $\Box$
Further complete examples 1

\[ T = \{ e : \varphi_e \text{ is total} \} \text{ is } \Pi^0_2 \text{-complete.} \]

**Proof.**

Recall \( T \) is \( \Pi^0_2 \). Let \( R \) be computable and

\[ A(x) \equiv \forall y \exists z \ R(x, y, z) \quad (\Pi^0_2) \]

- Define \( \psi(x, y) := \mu z \ R(x, y, z) \).
- By the \( S^m_n \)-Theorem for \( m = n = 1 \), we have a computable \( h := S^1_1 \) such that

\[ \psi(x, y) = \varphi_{h(x)}(y) \text{ for all } x, y. \]

- Then \( x \in A \text{ iff } \forall y \ \varphi_{h(x)}(y) \downarrow \text{ iff } \varphi_{h(x)} \text{ is total iff } h(x) \in T. \)

- Hence the \( S^m_n \)-Theorem yields a many-one reduction \( h \) from \( A \) to \( T \).
Further complete examples 2, 3

The **diagonal halting problem** $K = \{ x : \varphi_x(x) \downarrow \}$ is $\Sigma^0_1$-complete.

**Proof**

Let $R$ be computable and

$$A(x) \equiv \exists y \ R(x, y) \quad (\Sigma^0_1)$$

- Define $\psi(x, z) := \mu y \ R(x, y)$ (independent of $z$!).
- By the $S_n^m$-Theorem, we have a computable $h$ such that

$$\psi(x, z) = \varphi_{h(x)}(z) \text{ for all } x, z.$$

- Then $x \in A$ iff $\psi(x, z) \downarrow$
  iff $\varphi_{h(x)}(h(x)) \downarrow$
  iff $h(x) \in K$.

$K_n := \{ x : U_n(x, x) \}$ is $\Sigma^0_n$-complete for $n \geq 1$. 

$\square$