Arithmetical Hierarchy

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DTM vs functions on $\ensuremath{\mathbb{N}}$

For a partial function f write

- $f(x) \downarrow$ if x is in the domain of f;
- $f(x) \uparrow \text{ if } x \text{ is not in the domain of } f$.

 $\varphi_e(x) \colon \mathbb{N} o_p \mathbb{N}$ is computed by the DTM with Goedel number e

Facts

- A ⊆ N^k is computably enumerable iff A is the domain of some partial recursive function.
- A ⊆ N^k is computable iff the characteristic function of A is recursive.
- The Diagonal Halting Problem

$$\mathsf{K} := \{ x \in \mathbb{N} : \varphi_x(x) \downarrow \}$$

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is c.e. but not computable.

Properties of recursive functions are not computable

Rice's Theorem

Let *C* be a class of *k*-ary recursive functions. Then $\{e \in \mathbb{N} : \varphi_e \in C\}$ is computable iff $C = \emptyset$ or *C* is the class of all *k*-ary recursive functions.

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Example

None of the following are computable:

The arithmetical hierarchy of subsets of $\ensuremath{\mathbb{N}}$

Idea: Classify problems that are not computable by the complexity of formulas that describe them.

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Example (Diagonal halting problem K) $x \in K$ iff $\varphi_x(x) \downarrow$ iff $\exists y \quad (config(x, x, y))_0 = t$

computable predicate stating computation halts after y steps

Definition

Let $P(\bar{x})$ be a *k*-ary predicate on \mathbb{N} , $n \in \mathbb{N}$:

• *P* is Σ_n^0 if there is a computable predicate *R*:

$$P(\bar{x}) \equiv \underbrace{\exists y_1 \forall y_2 \exists y_3 \dots \exists / \forall y_n}_{n \text{ alternating quantifiers starting with } \exists : R(\bar{x}, \bar{y})$$

• *P* is Π_n^0 if there is a computable predicate *R*:

$$P(X) \equiv \underbrace{\forall y_1 \exists y_2 \forall y_3 \dots \exists / \forall y_n}_{: R(\bar{x}, \bar{y})}$$

n alternating quantifiers starting with \forall

•
$$\Sigma_0^0 = \Pi_0^0 = \text{computable predicates}$$

• $\Delta_n^0 := \Sigma_n^0 \cap \Pi_n^0$

Note

The superscript 0 denotes quantification over type-0-objects (elements in \mathbb{N}).

Example

1.
$$K \text{ is } \Sigma_1^0$$

2. $T = \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$
 $e \in T \text{ iff } \forall x \varphi_e(x) \downarrow$
 $\text{ iff } \forall x \exists y (\operatorname{config}(e, x, y))_0 = t$
Hence $T \text{ is } \Pi_2^0$.
3. $F = \{e \in \mathbb{N} : \varphi_e \text{ has finite domain}\}$
 $e \in F \text{ iff } \exists z \forall y \forall x (\operatorname{config}(e, x, y))_0 = t \Rightarrow x \leq z$
Hence $F \text{ is } \Sigma_2^0$.

$$\forall \omega \quad (\omega)_{0} = \gamma \quad (\omega)_{1} = \kappa$$

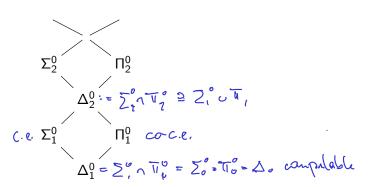
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Arithmetical hierarchy

 $\bigcup_{n \in \mathbb{N}} \Sigma_n^0 = \bigcup_{n \in \mathbb{N}} \prod_{n \in \mathbb{N}}^0$

(sets defined in first order arithmetic, hence called **arithmetical**)

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Question

Are all subsets of \mathbb{N} arithmetical?

For proving the previous picture we need some preparation.

Lemma

Let $n \ge 1$.

- 1. Σ_n^0 is closed under existential quantification, Π_n^0 is closed under universal quantification.
- Σ⁰_n, Π⁰_n are both closed under ∧, ∨, bounded quantifiers ∀x < y, ∃x < y, and substitution of total computable functions.

3.
$$\neg \Sigma_n^0 = \Pi_n^0, \neg \Pi_n^0 = \Sigma_n^0$$

Proof sketch.

Let R be computable, $P(x,z) = \exists y_1 \forall y_2 \dots R(\underline{x,z},y_1,\dots)$ be Σ_n^0 .

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1. Claim: $Q(x) := \exists z \ P(x,z)$ is Σ_n^0

$$Q(x) \equiv \exists z \exists y_1 \forall y_2 \dots R(x, z, y_1, y_2, \dots, y_n) \\ \equiv \exists u \quad \forall y_2 \dots R(x, (u)_0, (u)_1, y_2 \dots, y_n)$$

Dual argument for \forall and Π_n^0 .

2. Substitution: Let f(x) total, computable. Claim: Q(x) := P(x, f(x)) is Σ_n^0 $Q(x) \equiv \exists y_1 \forall y_2 \dots \underbrace{R(x, f(x), y_1, y_2, \dots)}_{\text{computable since } R \text{ is}}$

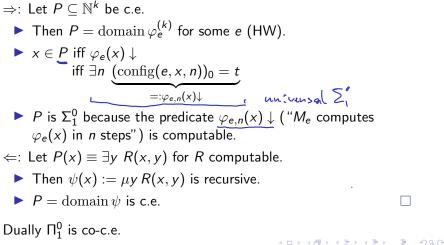
Conjunction: Induct on *n* (HW).

3. Negation: immediate.

Σ_1^0 is computably enumerable

Normal Form Theorem for c.e. sets P is c.e. iff P is Σ_1^0 .

Proof



Universal Σ_n^0 predicates

Idea: Enumerate k-ary Σ_n^0 predicates by a single k + 1-ary Σ_n^0 predicate.

Definition

A k + 1-ary $\sum_{n=0}^{n}$ predicate $U(e, \overline{x})$ is **universal** $\sum_{n=0}^{n}$ for k-ary predicates if

- 1. U(e, x) is Σ_n^0
- 2. for every k-ary $\sum_{n=0}^{0}$ predicate P(x) there is some e such that

$$P(x) \equiv U(e,x)$$

Universal Π_n^0 -predicates are defined similarly.

Example

From the last proof $U(e, x) := \exists n \varphi_{e,n}(x) \downarrow$ is universal Σ_1^0 .

Enumeration Theorem

For all $k, n \ge 1$, universal Σ_n^0 - and Π_n^0 -predicates exist.

Proof by induction on n and k:

Base case: $U(e, x) := \exists m \varphi_{e,n}(x) \downarrow$ is universal Σ_1^0 by the Normal Form Theorem for c.e. predicates.

Note: If U(e, x) is universal Σ_n^0 , then $\neg U(e, x)$ is universal Π_n^0 (and conversely).

Induction step: Let $U(e, y, \bar{x})$ be universal Σ_n^0 for k + 1-ary predicates. Then $\forall y \ U(e, y, \bar{x})$ is universal Π_{n+1}^0 for k-ary predicates since 1. it is in Π_{n+1}^0 and

2. for every k-ary Π_{n+1}^0 -predicate $P(\bar{x})$ there exists a k + 1-ary Σ_n^0 -predicate $Q(y, \bar{x})$ such that

The arithmetical hierarchy does not collapse

Corollary

For each $n \ge 1$ there exist Σ_n^0 -predicates that are not Π_n^0 (and conversely).

Proof.

- Let $U_n(e, x)$ a unary universal $\sum_{n=1}^{0}$ -predicate.
- Then $K_n(x) := U_n(x,x)$ is Σ_n^0 .
- Seeking a contradiction, suppose K_n is $\prod_{n=1}^{0}$. Then $\neg K$ is $\sum_{n=1}^{0}$.

- Hence $\neg K_n(x) \equiv U_n(e,x)$ for some *e*.
- Contradiction for x = e.