Normal Form Theorem for recursive functions

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Gödel numbering of DTM

To show that computable functions on $\mathbb{N}$ are recursive we first encode DTMs and their configurations as natural numbers.

- Let $2 = p_0 < p_1 < p_2 < \dots$ the enumeration of all primes.
- $\forall x \in \mathbb{N} \setminus \{0\}$ $\exists$ unique $x_0, x_1, \dots \in \mathbb{N}$, almost all 0:

$$x = \prod_{i \in \mathbb{N}} p_i^{x_i}$$

  The exponent $(x)_i$ of p_i in x is primitive recursive.
  - Recall $i \mapsto p_i$ is primitive recursive,
  - $(x)_i = \mu(t < x)\,[p_i^{t+1}$ does not divide $x]$ is primitive recursive.
- Encode a tuple $(a_0, \dots, a_n) \in \mathbb{N}^*$ as

$$a := p_0^{a_0+1} \cdots p_n^{a_n+1} \in \mathbb{N}.$$

  Then n and a_i for all $i \leq n$ are primitive recursive from a.

Definition

Let $M = (Q, \{0,1\}, \Gamma, s, t, r, \delta)$ be a DTM with

- n states $Q = \{1, \ldots, n\}$,
- tape alphabet $\Gamma = \{\gamma_0 = \sqcup, \gamma_1 = 1, \gamma_2 = 0, \gamma_4, \ldots, \gamma_m\}$,

We define

$$a_0 := 2^{|Q|}3^{|\Gamma|}5^s7^t11^r$$

and encode the i-th transition $\delta(q, \gamma_k) = (p, \gamma_\ell, d)$ for $1 \leq i \leq (n-2)m$ as

$$a_i := 2^q3^k5^p7^\ell11^{1+d}.$$

Then the **prime power encoding** of M is

$$\sharp(M) = \prod_{i=0}^{(n-2)m} p_i^{a_i}.$$

Definition

A configuration (q, α, k) of the DTM M is encoded as

$$\sharp(q, \alpha, k) := 2^q 3^{\prod_{i \in \mathbb{N}} p_i^{\alpha(i)}} 5^k$$

Note

- The encoding $\sharp(M)$ essentially IS (the transition function of) M.
- All constituents of $\sharp(M)$ and of the encoding of a configuration are primitive recursive.

Computable functions are recursive

Recall our encoding of tuples of natural numbers as strings:

- $n \in \mathbb{N}$ is represented as $\underbrace{1 \ldots 1}_{n+1}$
- (m, n) as $1^{m+1}01^{n+1}$

Definition

Let the partial function $\varphi_e^{(k)} \colon \mathbb{N}^k \to_p \mathbb{N}$ be computed by the DTM M with $\sharp(M) = e$.

Normal Form Theorem (Kleene)

For every $k \in \mathbb{N} \setminus \{0\}$ there exists a primitive recursive predicate $T^{(k)}(e, \bar{x}, y)$ and a primitive recursive function u such that

$$\varphi_e^{(k)}(\bar{x}) = u\big(\mu y\, T^{(k)}(e, \bar{x}, y)\big).$$

Proof sketch for $k = 1$.

Claim 1:

$$\mathrm{succ}(c) := \sharp(\text{successor of the configuration with encoding } c)$$

is primitive recursive.

Let c be the encoding of a configuration. Then

- $q := (c)_0$ is its state,
- $k := (c)_2$ is the position of the head,
- $i := ((c)_1)_k$ is the index of the cell content γ_i.

Then

$$\mathrm{succ}(e, c) := \begin{cases} 2^p 3^{(c)_1 \cdot p_k^{-i+j}} 5^{\max(k+d,0)} & \text{if } \delta(q, \gamma_i) = (p, \gamma_j, d), \\ c & \text{if } q \in \{t, r\}. \end{cases}$$

Note that p, j, d are primitive recursive from e, q, i. Claim 1 is proved.

Claim 2:

$$\mathrm{config}(e,x,n) := \sharp(\text{configuration of} M_e \text{ on input } x \text{ at step } n)$$

is primitive recursive by the recursion scheme:

$$\mathrm{config}(e,x,0) := 2^s 3^{\prod_{i\leq|x|} p_i^{x_i}} 5^0$$
$$\mathrm{config}(e,x,n+1) := \mathrm{succ}(\mathrm{config}(e,x,n))$$

Claim 3:

Assume M_e has computed $\varphi_e(x)$ iff it is in the accept state t. Then

$$\mathrm{halt}(e,x) := \mu n\,((\mathrm{config}(e,x,n))_0 = t)$$

is recursive and yields the number of steps for M_e to write $\varphi_e(x)$ on the tape and halt.

Then $\varphi_e(x)$ is primitive recursive from $\mathrm{config}(e,x,\mathrm{halt}(e,x))$. $\square$

Consequences of the Normal Form Theorem

- A partial function on $\mathbb{N}$ is computable (by a DTM) iff it is recursive (supporting the Church-Turing thesis again).
- Kleene's T-predicate and the function u in the Normal Form Theorem are universal, i.e., independent of the function φ_e.
- Recursive functions can be defined with at most one application of unbounded search μ.

Corollary (Enumeration Theorem)

$\psi(e, \bar{x}) := \varphi_e(\bar{x})$ is recursive.

Proof.

Immediate from the Normal Form Theorem. $\square$

Question

Why does the diagonalization argument not yield a computable function $d(x) := \varphi_x(x) + 1$ that is not recursive?

Dually to the Enumeration Theorem we have:

S_n^m-Theorem

For all $m, n \geq 1$ there exists a primitive recursive function S_n^m such that $\forall x \in \mathbb{N}, \bar{y} \in \mathbb{N}^m, \bar{z} \in \mathbb{N}^n$:

$$\varphi_{S_n^m(x,\bar{y})}(\bar{z}) = \varphi_x(\bar{y}, \bar{z})$$

Proof.

- Let M be the DTM that on input $\bar{z}$ simulates the DTM M_x on input $(\bar{y}, \bar{z})$.
- Then $\sharp(M) =: S_n^m(x, \bar{y})$ is primitive recursive from $x, \bar{y}$.

$\square$