Ackermann function

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Question

Primitive recursive functions are computable. What about the converse?

We'll see that some functions grow too fast to be primitive recursive.

Knuth's up arrow notation.

$a \uparrow^n b$ is defined by $a \uparrow b := \underbrace{a \cdots a}_{b}$

$$a \uparrow\uparrow b := \underbrace{a^{\cdot^{\cdot^{\cdot^{a}}}}}_{b}$$

$$a \uparrow^{n+1} b := \underbrace{a \uparrow^n (a \uparrow^n \ldots a)}_{b}$$

Definition

For $m, n \in \mathbb{N}$ define the **Ackermann function** $A(m,n)$ by

$$A(0,n) := n+1$$
$$A(m+1,0) := A(m,1)$$
$$A(m+1,n+1) := A(m, A(m+1,n))$$

(Not a primitive recursion scheme as it uses recursion over itself.)

Example

$$A(1,n) = n+2$$
$$A(2,n) = 2n+3$$
$$A(3,n) = 2^{n+3} - 3$$
$$A(4,n) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3} - 3$$
$$A(5,n) = 2 \uparrow\uparrow\uparrow (n+3) - 3$$

Facts

1. $A(m, n)$ is a total, computable function.
2. A is strictly increasing in each argument.
3. $A(m, n+1) \leq A(m+1, n)$
4. $A(\ell, A(m, n)) < A(\ell + m + 2, n)$

Proof ideas

1. Induction on (m, n) in lex order.
2. Induction on m, n respectively.
3. Induction on n.
4. $A(\ell, A(m, n)) < A(\ell + m, A(\ell + m + 1, n)) =$
   $A(\ell + m + 1, n + 1) \leq A(\ell + m + 2, n)$.

Majorization Lemma

For every primitive recursive $f(\bar{x})$ there exists $M \in \mathbb{N}$ such that

$$\forall \bar{x} \colon f(\bar{x}) < A(M, \max(\bar{x})).$$

Proof by induction on the representation of f.

Base cases $f = 0, s, p_i^k$ are straightforward for $M = 0, 1$.
Induction step:
1) **Composition:** Let $f(\bar{x}) := g(h_1(\bar{x}, \ldots, h_n(\bar{x})))$ for $g, h_1, \ldots, h_n$ primitive recursive.
Let $x := \max(\bar{x})$. By induction assumption we have $G, H \in \mathbb{N}$ such that

$$g(\bar{y}) < A(G, y), \;\; h_i(\bar{x}) < A(H, x) \text{ for all } i.$$

Then

$$f(\bar{x}) < A(G, \max(h_i(\bar{x}))) < A(G, A(H, x))) < A(G + H + 2, x).$$

2) **Recursion scheme:** Let $f(\bar{x}, y)$ be defined by

$$f(\bar{x}, 0) := g(\bar{x})$$
$$f(\bar{x}, y+1) := h(\bar{x}, y, f(\bar{x}, y))$$

for g, h primitive recursive.

By induction assumption we have $G, H \in \mathbb{N}$ such that

$$g(\bar{x}) < A(G, x),\ \ h(\bar{x}, y, z) < A(H, \max(x, y, z)).$$

Claim: $f(\bar{x}, y) < A(F, x+y)$ for $F := \max(G, H) + 1$ $\quad (\dagger)$

Induct on y:

Base case:

$$f(\bar{x}, 0) = g(\bar{x}) < A(G, x) < A(F, x)$$

Induction step:

$$f(\bar{x}, y+1) = h(\bar{x}, y, f(\bar{x}, y)) < A(H, \max(x, y, f(\bar{x}, y)))$$

By the induction hypothesis and $x, y < A(F, x+y)$,

$$\max(x, y, f(\bar{x}, y))) < A(F, x+y).$$

Now Claim $(\dagger)$ follows from

$$f(\bar{x}, y{+}1) < A(H, A(F, x{+}y)) \leq A(F{-}1, A(F, x{+}y)) = A(F, x{+}y{+}1).$$

Finally let $z := \max(x, y)$. Using Claim (†)

$$f(\bar{x}, y) < A(F, 2z) < A(F, 2z + 3) = A(F, A(2, z)) < A(F + 4, z).$$

The Majorization Lemma is proved. □

Corollary

The Ackermann function $A(m, n)$ is not primitive recursive.

Proof.

Seeking a contradiction, suppose otherwise.

- Then $f(n) := A(n, n)$ is primitive recursive.
- By the Majorization Lemma we have $M \in \mathbb{N}$ such that $f(n) < A(M, n)$.
- Then $f(M) < A(M, M)$ is a contradiction.

□