Deterministic and nondeterministic automata

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Recall

Definition

A **deterministic finite automaton (DFA)** is a 5-tuple $M = (Q, \Sigma, \delta, s, F)$ with

- Q a finite set (**states**),
- Σ a finite set (**input alphabet**),
- $\delta\colon Q \times \Sigma \to Q$ the **transition function**,
- $s \in Q$ the **start state**,
- $F \subseteq Q$ the set of **final/accepting states**.

M **accepts** $w \in \Sigma^*$ if $\delta^*(s, w) \in F$ for the extension δ^* of δ to Σ^*.

$$L(M) := \{w \in \Sigma^* \ : \ M \text{ accepts } w\}$$

is the **language of** M.

Example

Is there a DFA M_3 such that
$L(M_3) = \{w \in \{0,1\}^* \ : \ 001 \text{ is a substring of } w\}$?

Idea: Scan w for 001

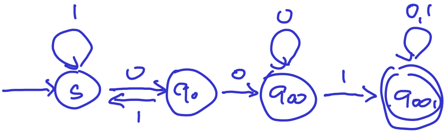
States: start s ... no part of 001 seen yet

q_0 ... just saw 0

q_{00} ... just saw 00

q_{001} ... 001

Graphical representation



Nondeterministic finite automata

- **Deterministic:** current state and input symbol uniquely determine next state
- **Nondeterministic:** several choices for next state
  - Interpretation: all possible transitions are done in parallel/the 'right' one is guessed.
  - Not a realistic model of computation but a useful theoretical device for its analysis.

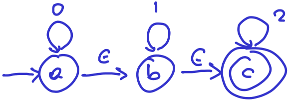
Definition

A **nondeterministic finite automaton with ϵ-transitions (ϵ-NFA)** is a 5-tuple $(Q, \Sigma, \Delta, s, F)$ like a DFA except that

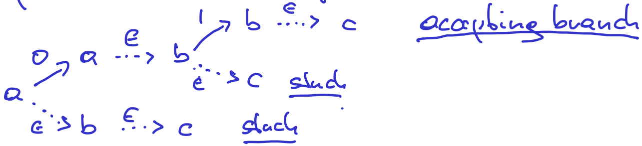
$$\Delta\colon Q \times \Sigma \cup \{\epsilon\} \to P(Q) \qquad (P(Q) \ldots \text{power set of } Q)$$

- Recall ϵ is the empty word, not an element in Σ.
- ϵ-transitions allow the NFA to change from a state q to any state in $\Delta(q, \epsilon)$ without input.
- Wlog, the sets $T := \Delta(q, a)$ for any $a \in \Sigma \cup \{\epsilon\}$ are **ϵ-closed** (i.e. if $t \in T$, then also $\Delta(t, \epsilon) \subseteq T$).

Example ϵ-NFA N



Computational tree on input: 01



$$\Delta^*(a, 01) = \{b, c\}$$

final state

$$\Delta^*(a, 10) = \phi$$

$$L(N) = \{0^l 1^m 2^n : l, m, n \in \mathbb{N}\}$$

Definition

For an ϵ-NFA $N = (Q, \Sigma, \Delta, s, F)$ the **extended transition function**

$$\Delta^*\colon Q \times \Sigma^* \to P(Q)$$

is defined inductively for $q \in Q, w \in \Sigma^*, a \in \Sigma$ by

$$\begin{array}{ll} \Delta^*(q, \epsilon) & := \Delta(q, \epsilon) \\ \Delta^*(q, wa) & := \bigcup_{r \in \Delta^*(q,w)} \Delta(r, a) \end{array}$$

assuming all $\Delta(q, \epsilon)$ and $\Delta(r, a)$ are ϵ-closed.

- N **accepts** w if $\Delta^*(s, w) \cap F \neq \emptyset$
  (i.e. N accepts w iff $\exists$ some path from s to a state in F that is labelled by w).
- N **rejects** w otherwise.

The **language of** N is

$$L(N) := \{w \in \Sigma^* \ : \ M \text{ accepts } w\}.$$

Note

- Every DFA can be considered as ϵ-NFA with $\Delta(q, a) := \{\delta(q, a)\}$ (singleton) and $\Delta(q, \epsilon) := \emptyset$.
- Every language accepted by a DFA is also accepted by some ϵ-NFA. What about the converse?

Theorem (Subset construction (Rabin, Scott 1959))

Let $N = (Q, \Sigma, \Delta, s, F)$ be an ϵ-NFA with all $\Delta(q, a)$ ϵ-closed. Let $M = (Q', \Sigma, \delta, s', F')$ be the DFA with

- $Q' := P(Q)$,
- $\delta(R, a) := \bigcup_{q \in R} \Delta(q, a)$ for $R \subseteq Q, a \in \Sigma$,
- $s' := \Delta(s, \epsilon)$,
- $F' := \{R \in P(Q) \ : \ R \cap F \neq \emptyset\}$.

Then $L(N) = L(M)$.

Proof

First show for all $w \in \Sigma^*$ that

$$\delta^*(s', w) = \Delta^*(s, w) \qquad (\dagger)$$

by induction on $|w|$.

Base case: For $w = \epsilon$,

$$\delta^*(s', \epsilon) \overset{\text{Def } \delta^*}{=} s' \underset{\text{Def } s'}{=} \Delta(s, \epsilon) \overset{\epsilon\text{-closed}}{=} \Delta^*(s, \epsilon).$$

Induction hypothesis: $(\dagger)$ holds for $w \in \Sigma^*$ of length n.

Let $a \in \Sigma$. Then

$$\begin{aligned}
\delta^*(s', wa) &= \delta\big(\delta^*(s', w), a\big) && \text{by definition of } \delta^* \\
&= \delta\big(\Delta^*(s, w), a\big) && \text{by induction hypothesis} \\
&= \textstyle\bigcup_{q \in \Delta^*(s,w)} \Delta(q, a) && \text{by definition of } \delta \\
&= \Delta^*(s, wa) && \text{by definition of } \Delta^*
\end{aligned}$$

Hence $(\dagger)$ is proved.

Proof, continued

Note: N accepts w iff $\Delta^*(s, w) \cap F \neq \emptyset$
iff $\delta^*(s', w) \in F'$ by (†) and the definition of F'
iff M accepts w. $\square$

Note

The subset construction translates an NFA with $|Q|$ states into a DFA with $2^{|Q|}$ states. Often fewer suffice.

Example, continued

Recall the ϵ-NFA N with 3 states, $L(N) = \{0^\ell 1^m 2^n \ : \ \ell, m, n \in \mathbb{N}\}$. There is a DFA M with $L(M) = L(N)$ and

δ	0	1	2
$\{a, b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{c\}$
$\{b, c\}$	$\emptyset$	$\{b, c\}$	$\{c\}$
$\{c\}$	$\emptyset$	$\emptyset$	$\{c\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Since the other subsets cannot be reached from the starting state $\{a, b, c\}$, they can be omitted.